Motion on a line

$x(t)$, $\frac{d}{dx} x(t) = f(x(t))$;

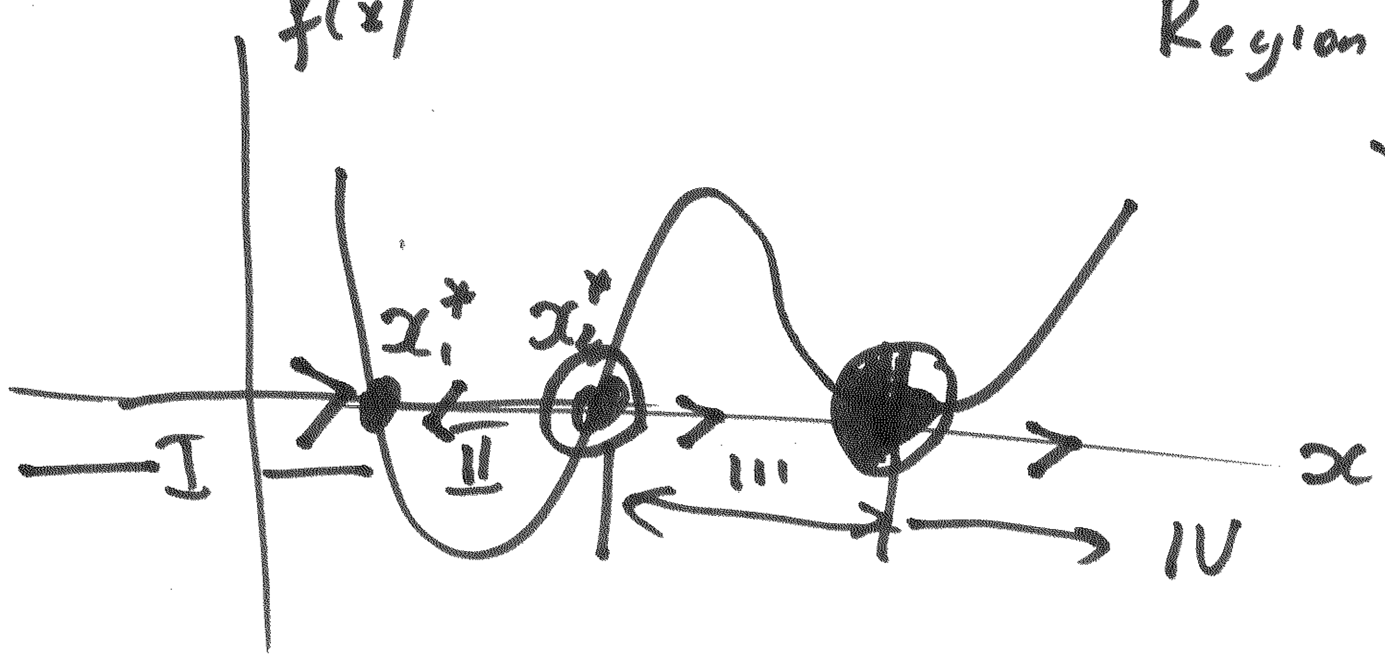
1st order, nonlinear, autonomous $\cancel{f(x(t), t)}$

Fixed point, critical point, equilibrium x^* such that

$f(x^*) = 0$

IF Initial condition $x(t=0) = x^*$ then $x(t) = x^*$ $\forall t > 0$

Plot $f(x)$ as a function of x



Region I $f(x) > 0$, $x(t)$ increases

III, IV — " —

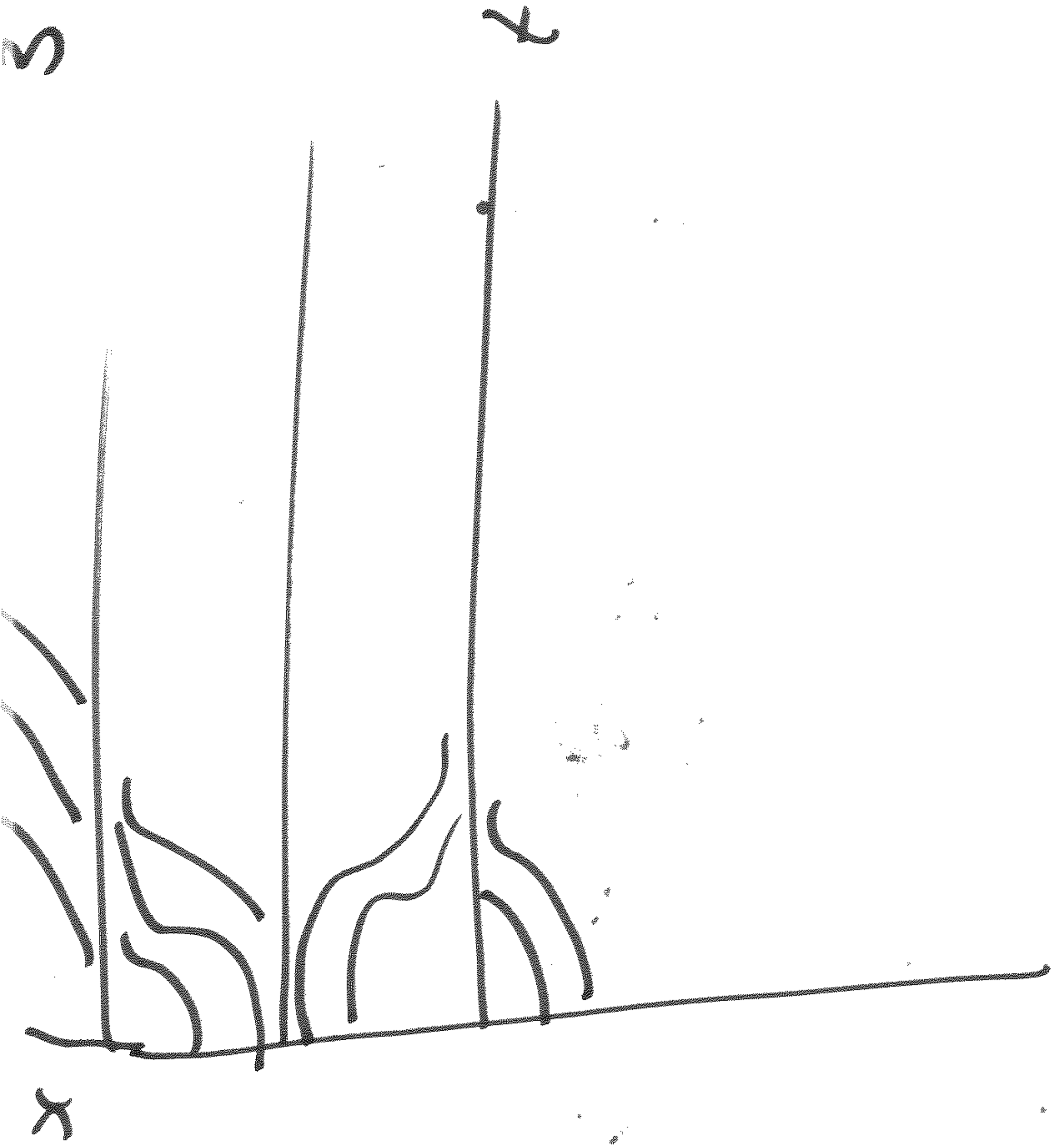
II $f(x) < 0$

$x(t)$ decreases

Stable fixed point x_1^*

unstable x_2^*

Semistable x_3^*


t
x

Linear Stability analyses

$$\dot{x}(t) = f(x(t));$$

$$f(x^*) = \emptyset;$$

$$x(t) = x^* + \eta(t);$$

$$\dot{\eta}(t) = f(x^* + \eta(t)) \approx$$

$$\approx f(x^*) + f'(x^*)\eta(t) + f''(x^*)\eta^2/2 + f'''(x^*)\eta^3/6 + \ldots$$

$f(x^*) = \emptyset$: $f'(x^*) \neq \emptyset$: $\dot{\eta}(t) = f'(x^*)\eta(t)$

5

$\dot{\eta}(t) = f'(x^*)\eta(t);$

$$\eta(t) = \eta(t=0)\, e^{f'(x^*)t}$$

$f'(x^*) > 0$, unstable

$f'(x^*) < 0$, stable

IF $f'(x^*) = 0$, $f''(x^*) \neq 0$;

$$\frac{d}{dt}\eta(t) = f''(x^*)\,\frac{\eta^2(t)}{2}$$ semistable

Logistic Equation

$$\dot{x}(t) = x(t)(1 - x(t));$$

$$\dot{x}(t) = a\,x(t)(b - x(t))$$

$$t = \alpha\,\tau; \quad [t] = \text{time}$$

$$[\alpha] = \text{time}$$

$$[\tau] = 1$$

$$x = \beta\,y(t)$$

$$\frac{d}{dt} = \frac{d}{d\tau}\,\frac{1}{\alpha}$$

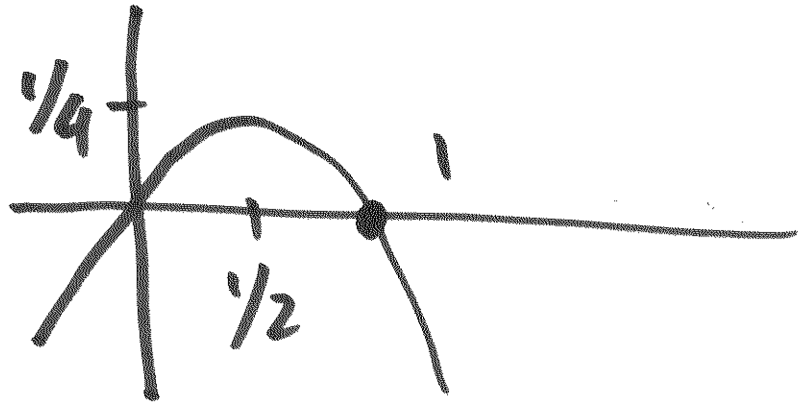
$$\frac{1}{\alpha}\,\frac{d}{d\tau}\,\beta\, y(\tau) = a\,\beta\, y(\tau)\,(b - \beta\, y(\tau))$$

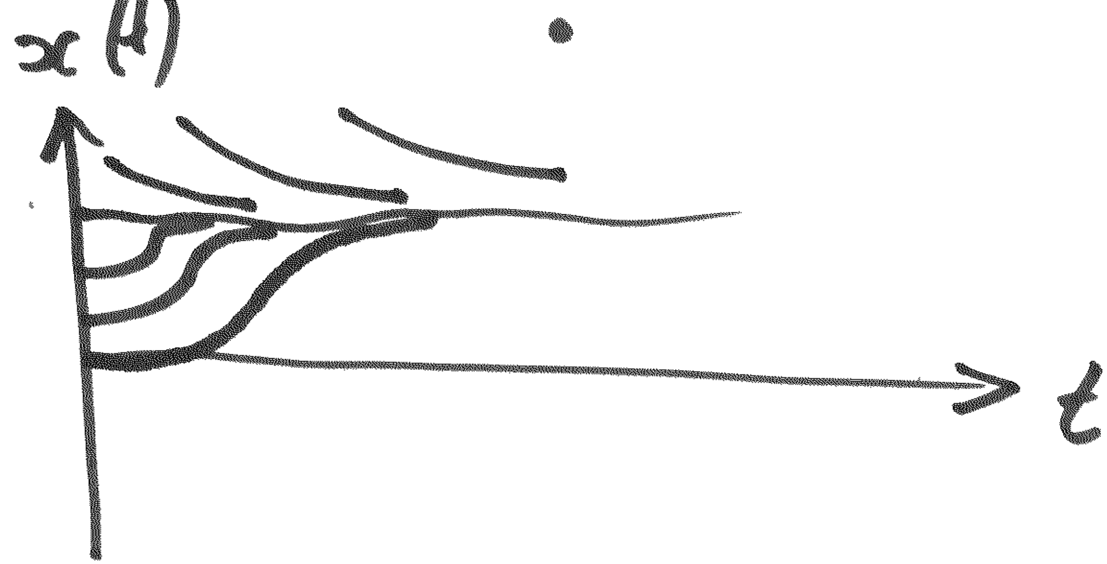
$$\frac{d}{d\tau}\, y(\tau) = a\,\alpha\, b\, y(\tau)\left(1 - \frac{\beta}{b}\, y(\tau)\right)$$

Choice $\beta = b$, $\alpha = 1/(a\,b)$

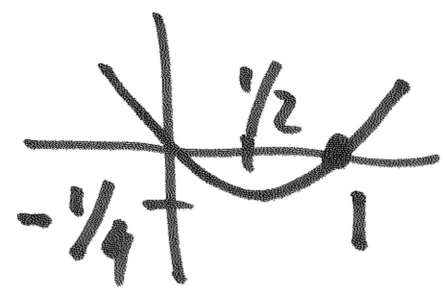
$$\frac{d}{d\tau}\, y(\tau) = y(\tau)\,(1 - y(\tau))$$

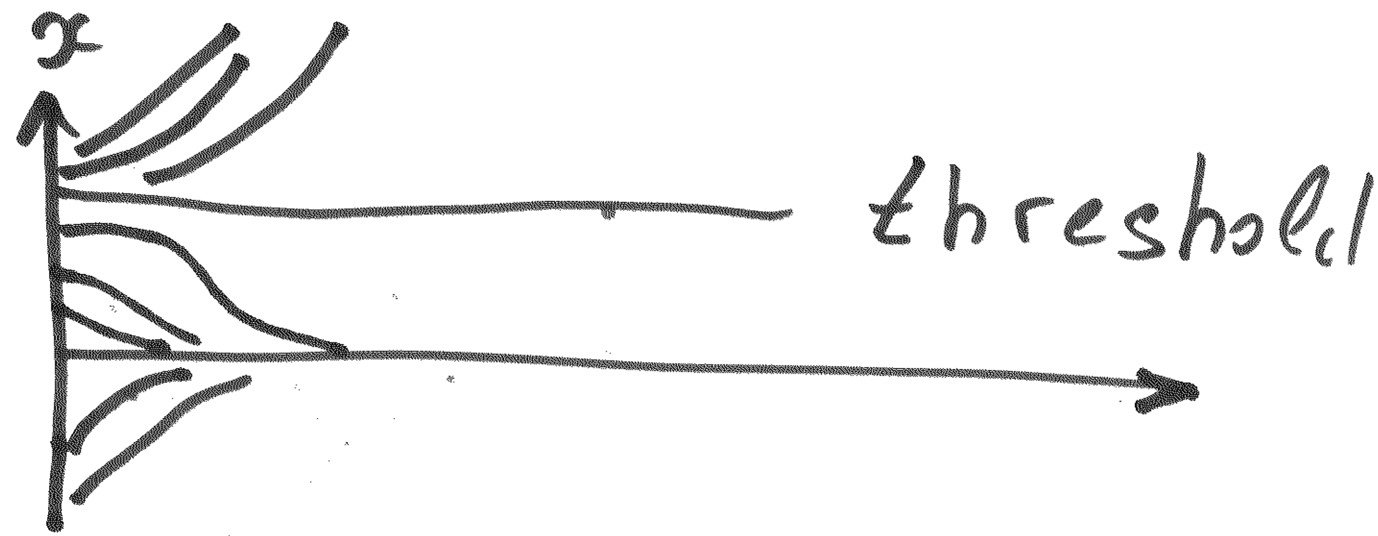
$$\dot{x} = x(1-x)$$



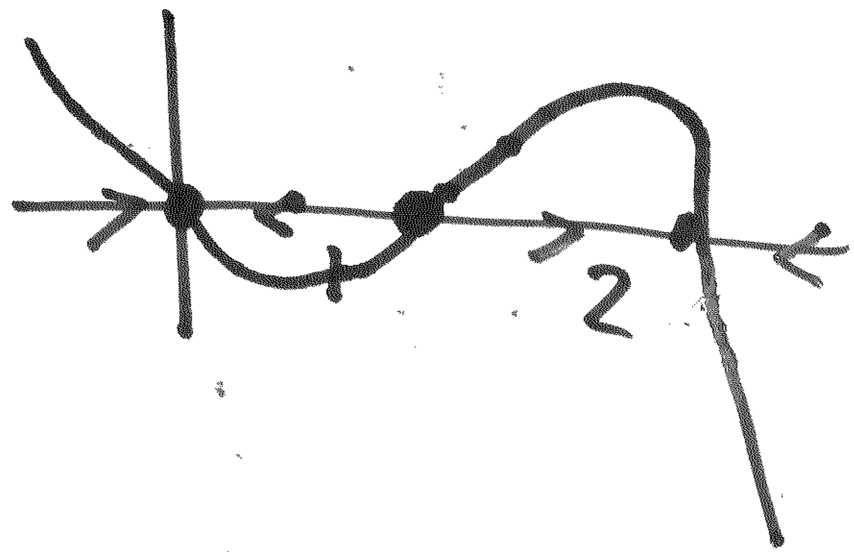


$$\dot{x} = -x\,(1-x) = x(x-1)$$





$$\dot{x} = x(x-1)(2-x)$$



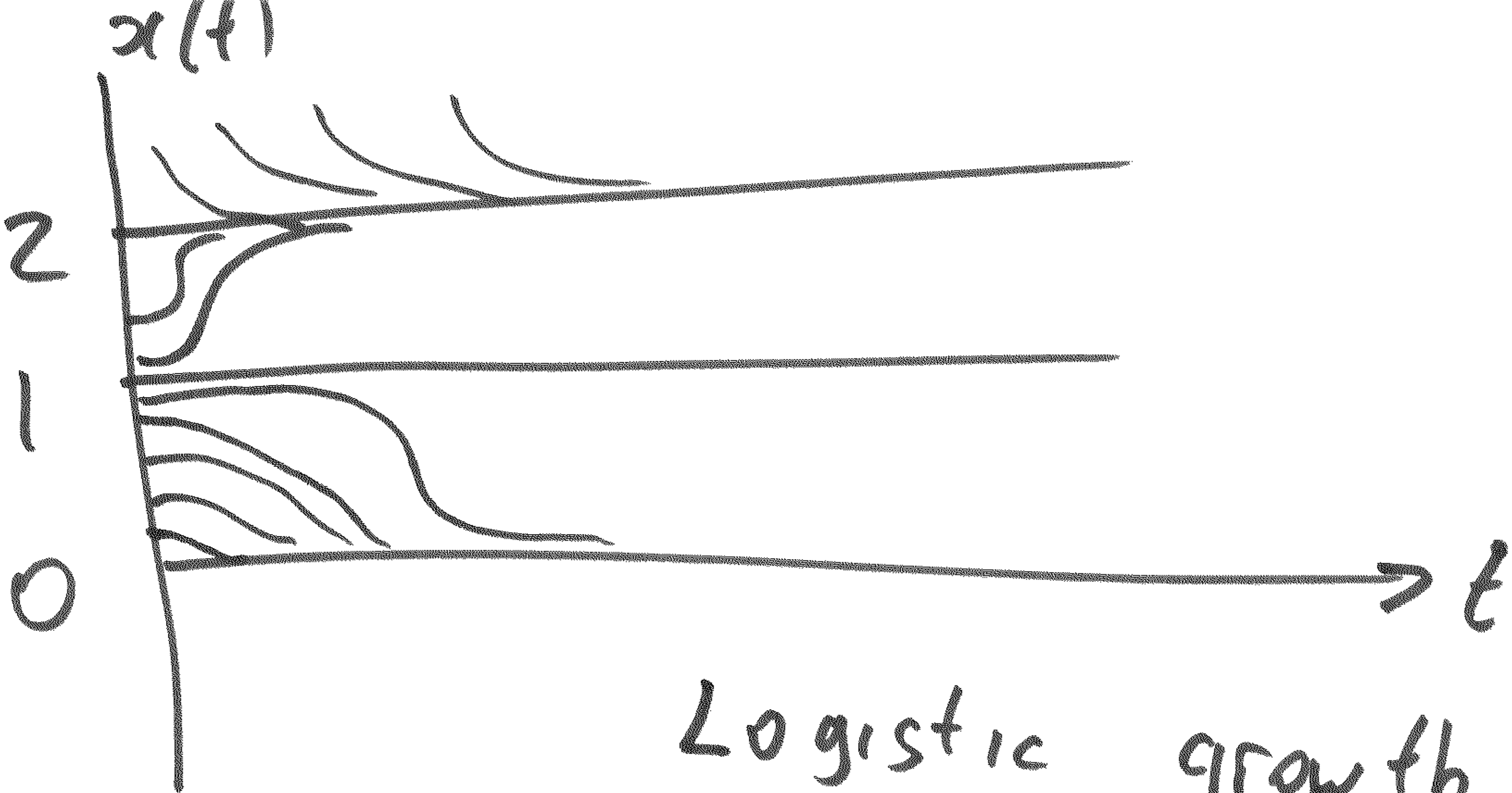
$x = 0$, stable
$x = 1$ unstable
$x = 2$ stable

$$f(x) = x(x-1)(2-x) = x(3x - x^2 - 2)$$

$$= -x^3 + 3x^2 - 2x$$

$$f'(x) = -3x^2 + 6x - 2;$$

$x = 0$, $f'(0) = -2$
$x = 1$, $f'(1) = 1$
$x = 2$ $f'(2) = -2$



Logistic growth with a treshold

Potential

$$\dot{x}(t) = f(x(t)) = -\frac{d\,\mathcal{V}(x)}{dx}$$

$$\mathcal{V}(x) = -\int f(x)\,dx$$

$$\frac{d}{dt}\mathcal{V}(x(t)) = \frac{d\,\mathcal{V}(x)}{dx}\,\frac{d\,x(t)}{dt} =$$

$$= -\frac{d\mathcal{V}(x)}{dx}\,\frac{d\mathcal{V}}{dx} = -\left(\frac{d\mathcal{V}(x)}{dx}\right)^2 \leq 0$$

Bifurcations.

saddle node

transcritical

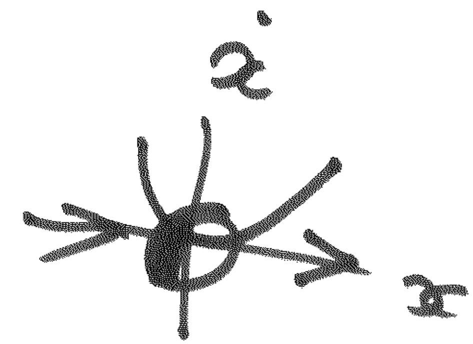
pitchfork

Saddle node

$\dot{x} = r + x^2$, normal form

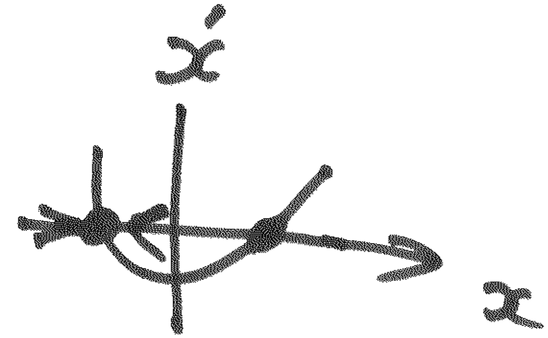
$r > 0 \Rightarrow$ no fixed point

$r = 0 \Rightarrow x = 0$, semistable



$r < 0$: $\dot{x} = -|r| + x^2 \Rightarrow$

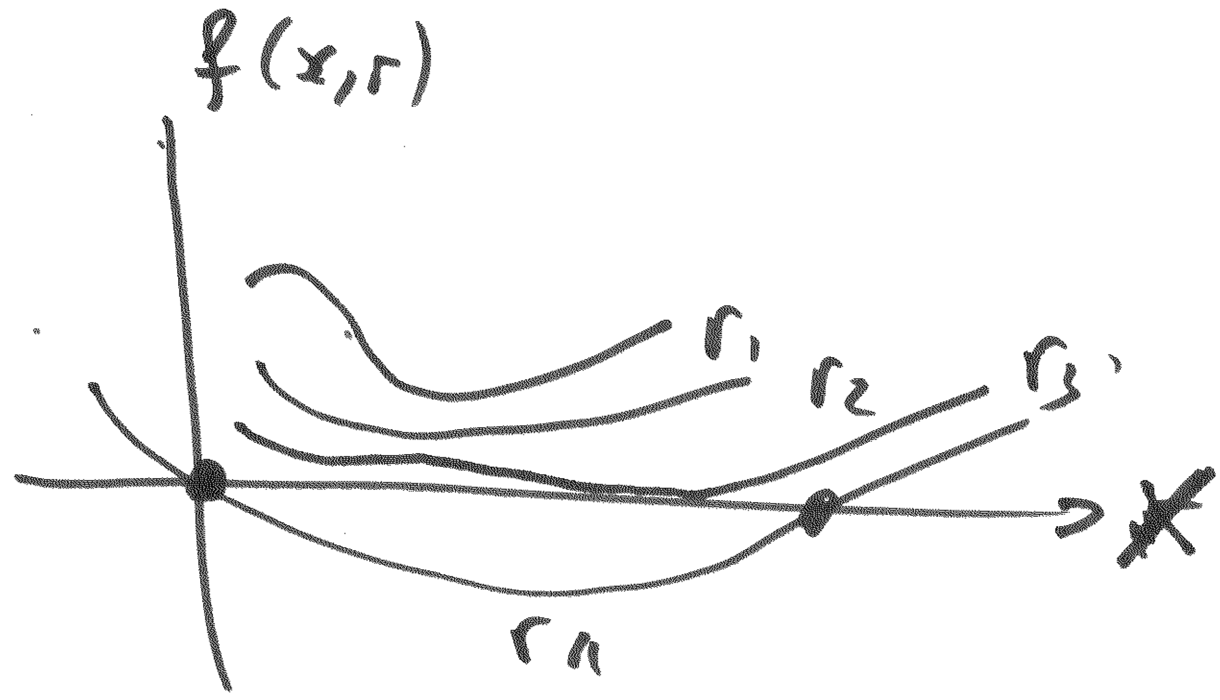
$x_* = \pm\sqrt{-r}$



$\frac{df}{dx} = 2x$

$x^* = \sqrt{-r}$ unstable

$x^* = -\sqrt{-r}$ stable



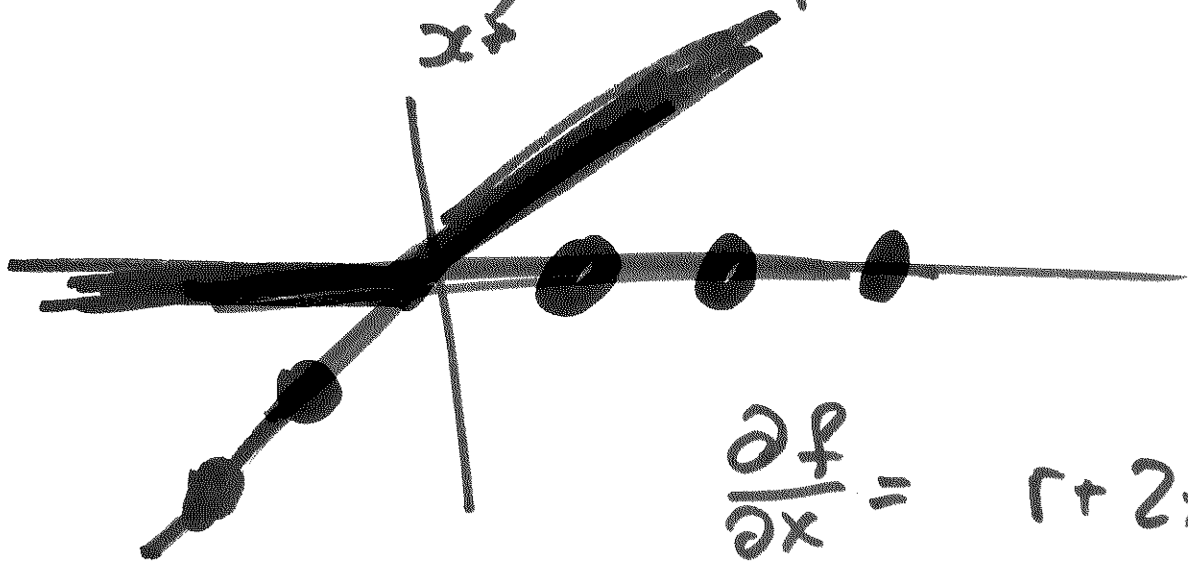
blue sky bifurcation

Transcritical Bifurcation (laser)

$$\dot{x} = x(r+x) = xr + x^2 = f(x,r)$$

$x = \varnothing, \; x = -r$

x^*



$$\frac{\partial f}{\partial x} = r + 2x$$

$x = \varnothing$: $\frac{\partial f}{\partial x} = r$, $\quad r > \varnothing$ unstable, $r < \varnothing$ stable

$x = -r$: $\frac{\partial f}{\partial x} = -r$ $\quad r > \varnothing$ stable, $r < \varnothing$ unstable

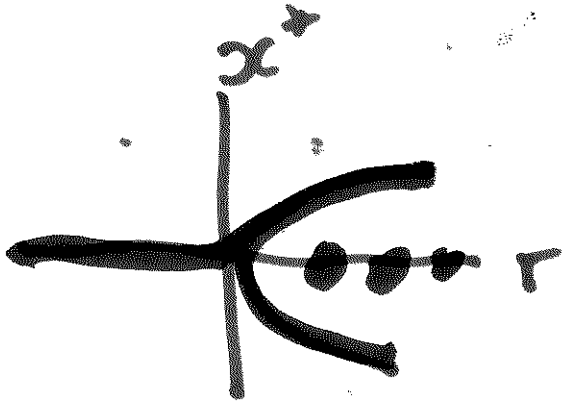
Supercritical pitchfork

$$\dot{x} = x(r - x^2) = rx - x^3$$

$$x = \emptyset, \quad x = \pm\sqrt{r}$$

$$\frac{\partial f}{\partial x} = r - 3x^2;$$

$x = \emptyset$: stable, $r < \emptyset$
unstable, $r > \emptyset$



$$x = \pm\sqrt{r};$$

$$\frac{\partial f}{\partial x} = r - 3r = -2r,$$ stable $r > \emptyset$
unstable $r < \emptyset$

(g)

$\theta = \phi$: stable, $\ell < a$

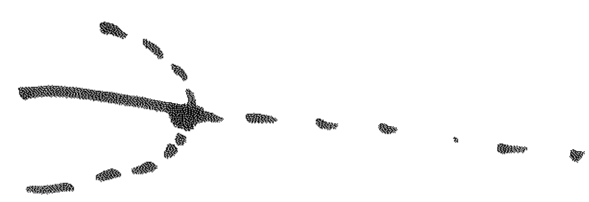
unstable, $\ell > a$

— $\dot{x} = r + x^2$

$\dot{x} = x(r+x)$

$\dot{x} = x(r-x^2)$

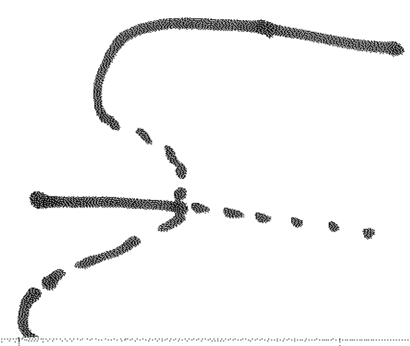
Subcritical pitchfork: $\dot{x} = x(r+x^2)$

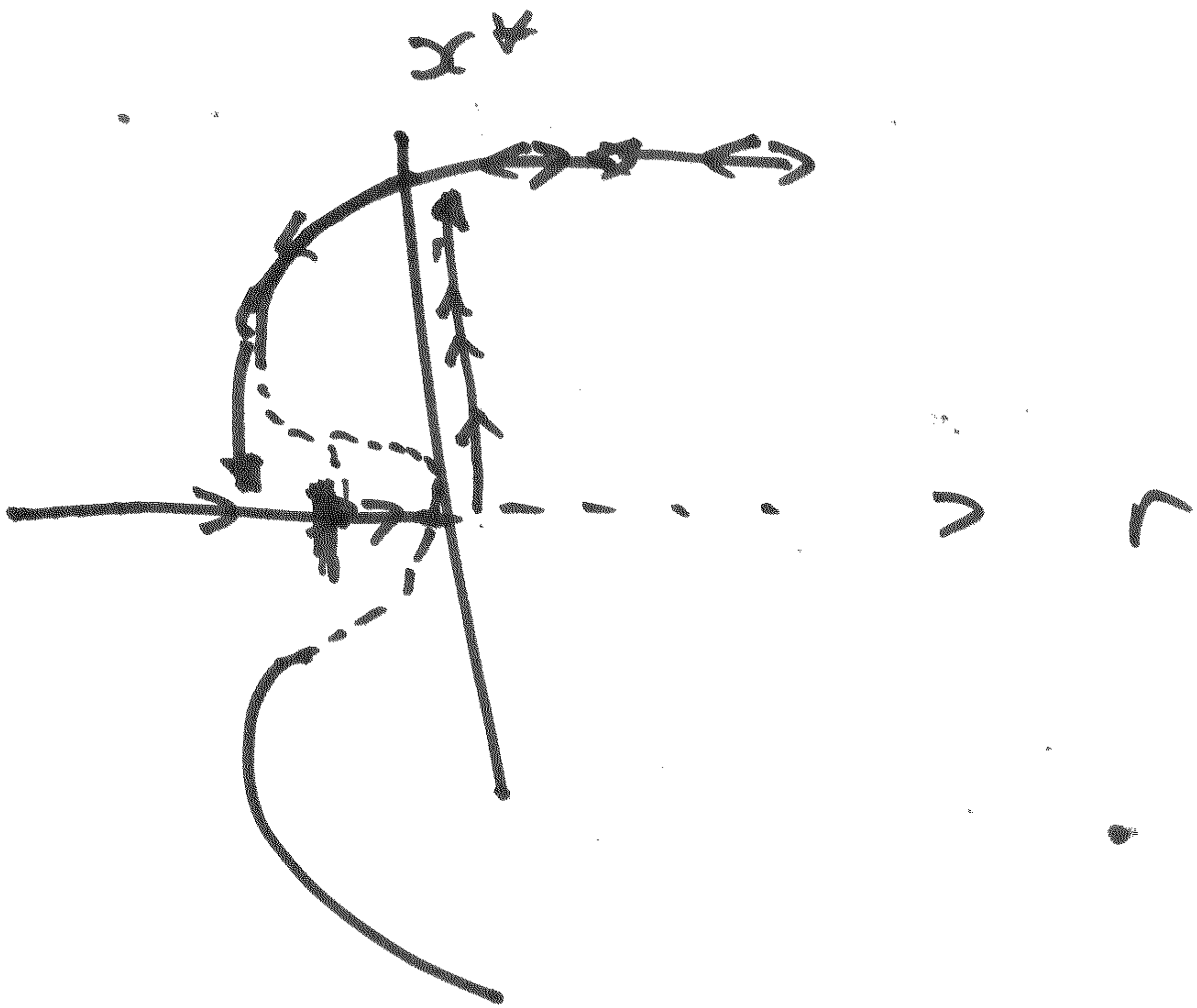


Modified subcritical

$\dot{x} = x(r+x^2 - x^4)$

↑ stabizer




x^*

Motion on a circle

$\dot{x} = f(x)$

$$\theta = x \bmod 2\pi$$

$$f(x + 2\pi) = f(x);$$

$f(x)$ is continious

$\dot{\theta} = f(\theta)$, $\quad \theta \equiv \theta \bmod 2\pi$

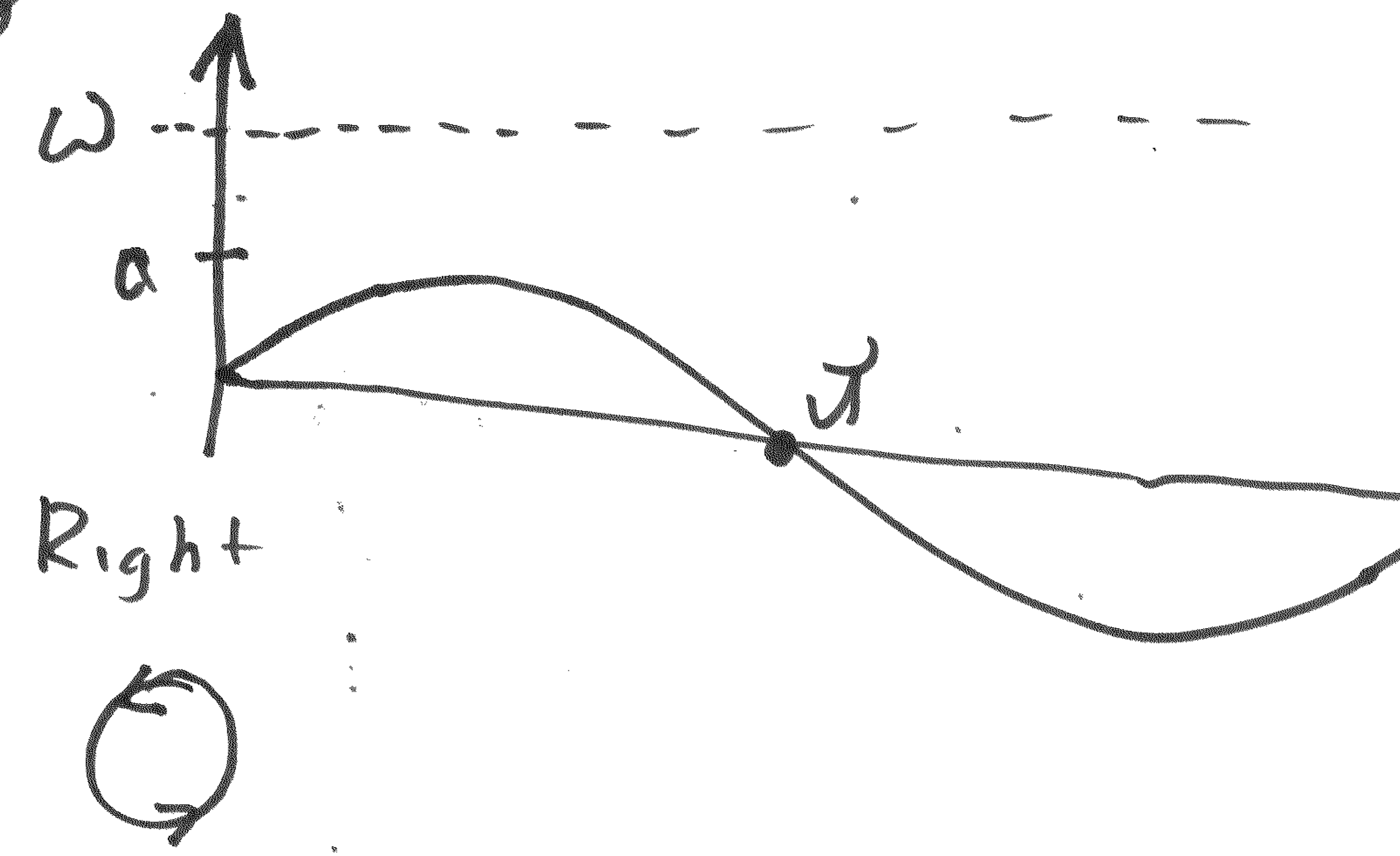
$$f(\theta + 2\pi) = f(\theta)$$

$$\dot{\theta} = \sin(\theta + 2\pi \sin\theta)$$

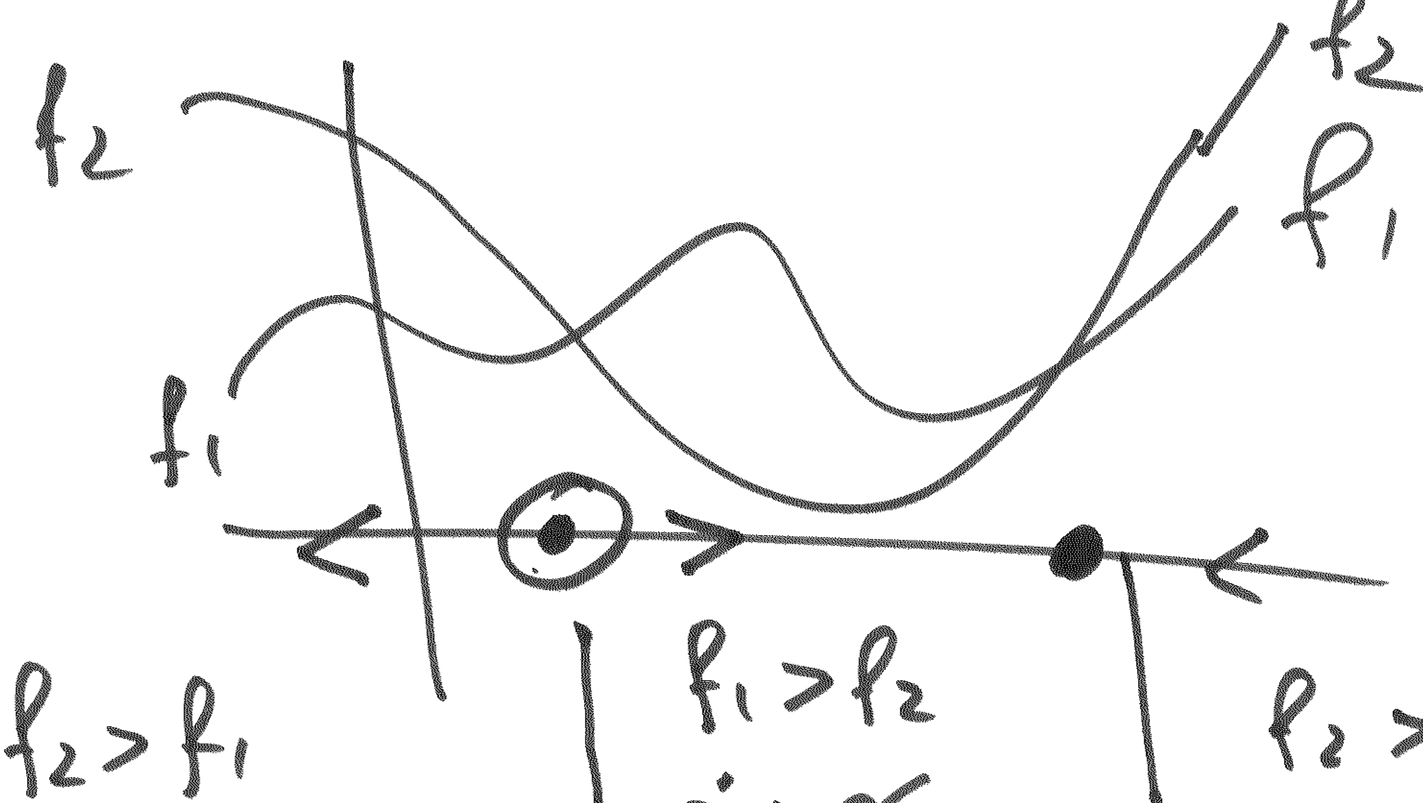
nonlinear oscillator

$\dot{\theta}(t) = \omega - a \sin\theta(t) \qquad \omega, a \geq 0$

- $a = 0$: $\dot{\theta}(t) = \omega \Rightarrow \theta(t) = \omega t + \theta(t=0)$
- $\omega > a > 0$



$$\dot{x}(t) = f_1(x(t)) - f_2(x(t))$$



$f_2 > f_1$

$\dot{x} < 0$, left

$f_1 > f_2$

$\dot{x} > 0$

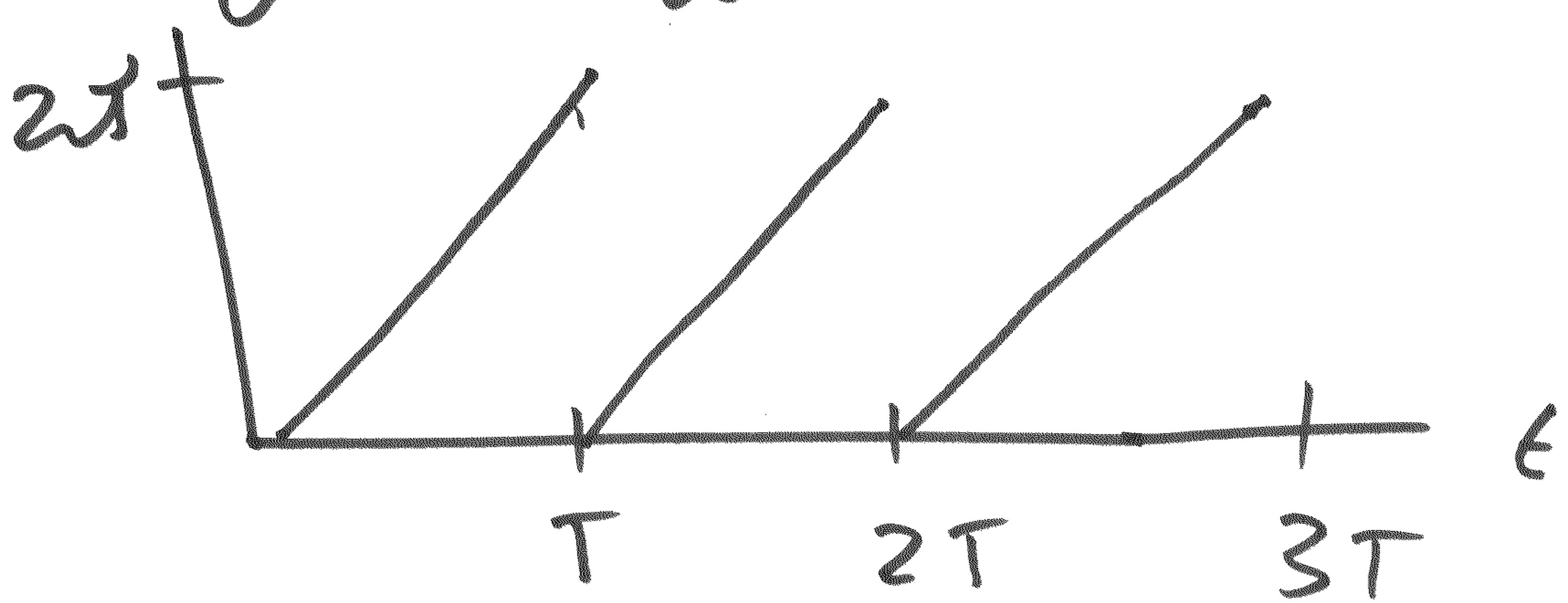
Right

$f_2 > f_1$

$\dot{x} < 0$

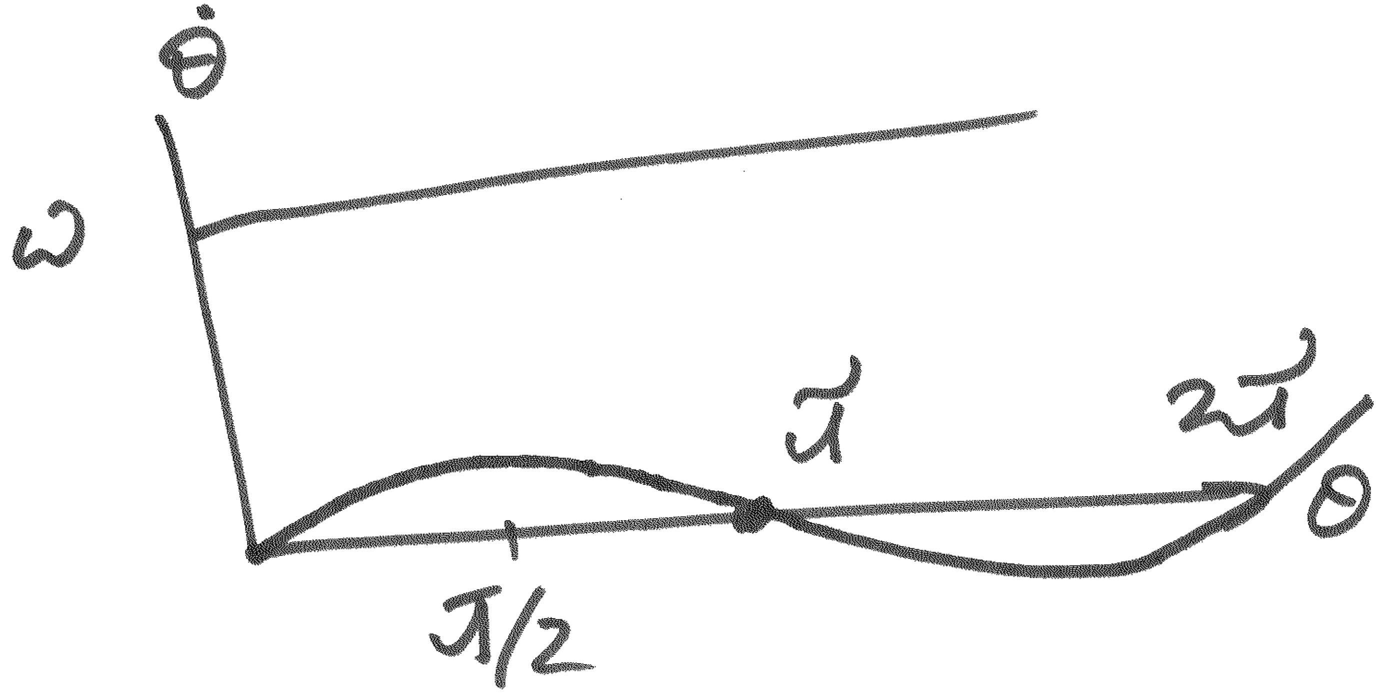
Left

$a = 0$, $T = \frac{2\pi}{\omega}$



uniform
oscillator

$$\dot{\theta} = \omega - a \sin \theta$$



$\dot{\theta} = \omega - a \sin\theta$

$a = \omega - \varepsilon, \quad \cancel{0} < \varepsilon \ll 1$