Lasing threshold

$$\dot{n}(t) = (G N_0 - k)\, n(t) - \alpha G n^2(t);$$

G, k, α are parameters of laser

N_0 - varying external parameter

G - gain coefficient

k — inverse life of a photon

α

$n = \emptyset$ - no lasing

$$G N_0 - k = \alpha G n,$$

$$n = \frac{G N_0 - k}{\alpha G}$$

$$\dot{n}(t) = f(n)$$

$$\frac{df}{dn} = \frac{d}{dn}\left((G N_0 - k) n - \alpha G n^2 \right)$$

$$= G N_0 - k - 2\alpha G n;$$

$\underline{n = \emptyset}$: $\dfrac{df}{dn} = G N_0 - k$, $\quad G N_0 - k < \emptyset$ stable

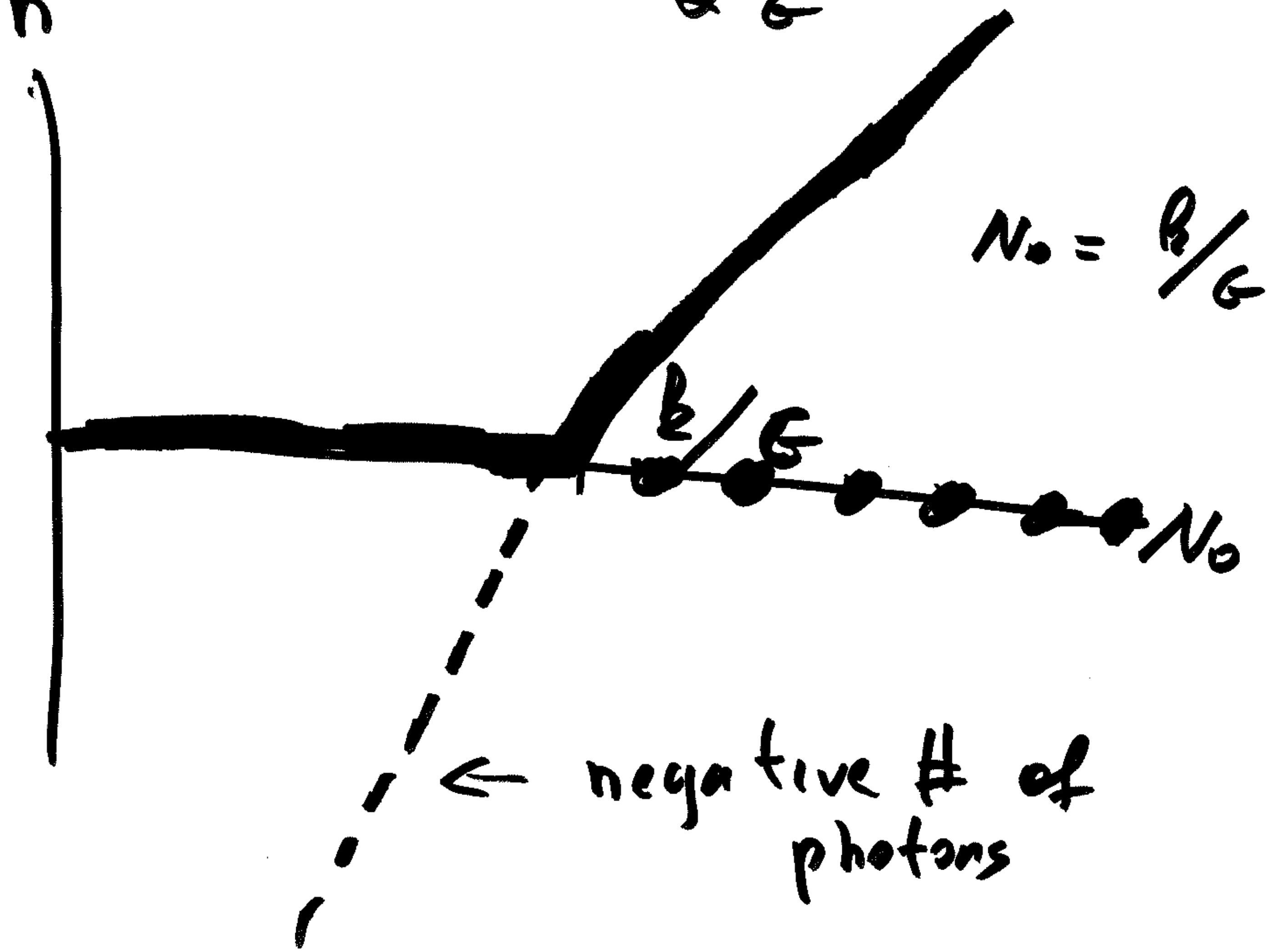
$\qquad G N_0 - k > \emptyset$ - unstable

$n = \dfrac{G N_0 - k}{\alpha G}$: $\dfrac{df}{dn} = G N_0 - k - 2(G N_0 - k) = - G N_0 + k$

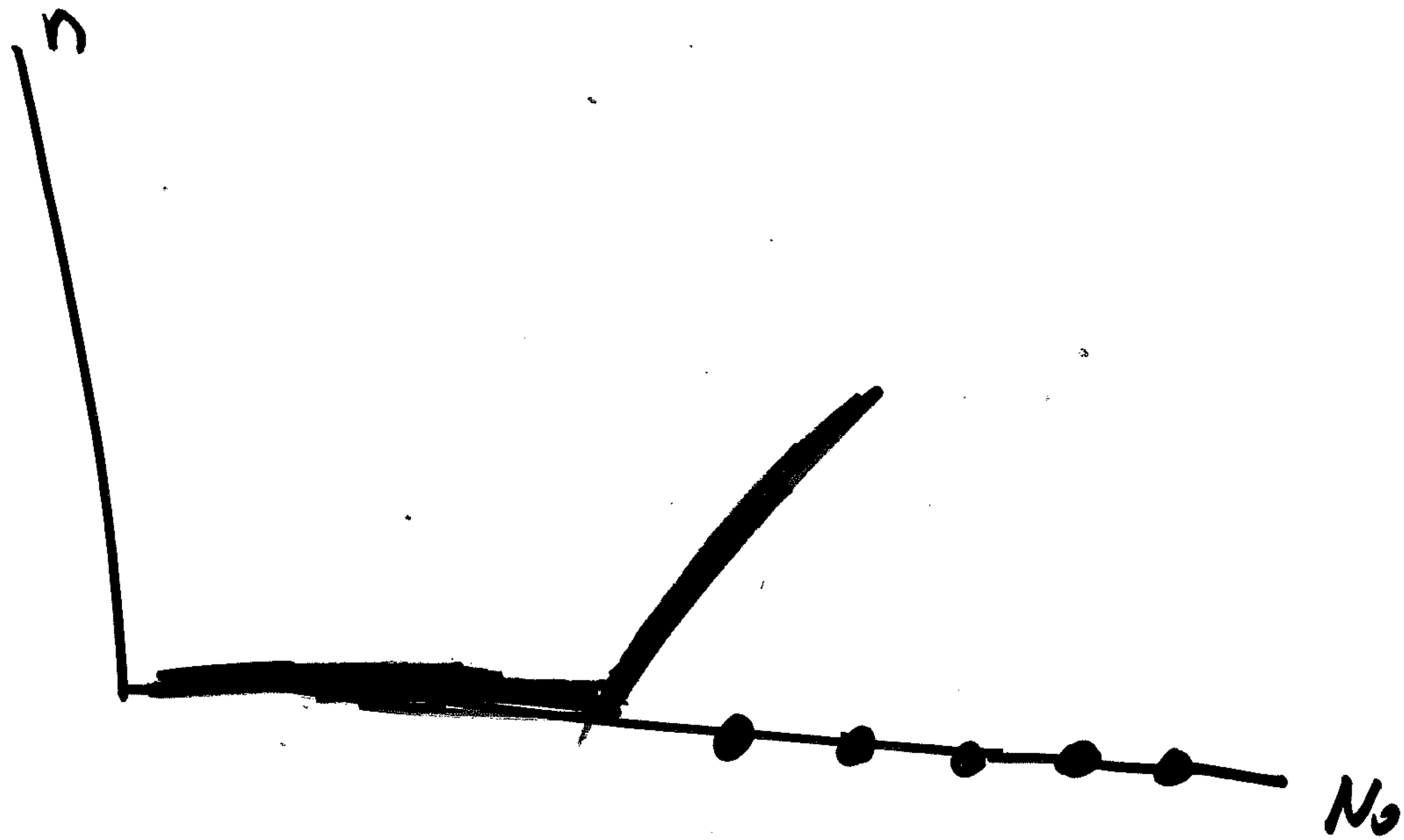
$$\dot{n} = (G N_0 - k)\,n - 2\,G n^2$$

$$n = \emptyset, \quad n = \frac{G N_0 - k}{2G}$$

<u>math</u>



physics



Supercritical Pitchfork Bifurcation. -5-

$\dot{x}(t) = r\,x(t) - x^3(t)$

-6-

Fixed points: $x^* = \emptyset$;

$r = x_*^2, \quad x^* = \pm\sqrt{r}$

$\dot{x} = f(x,r) = rx - x^3$

$\frac{\partial f}{\partial x} = r - 3x^2$; $\quad x^* = \emptyset$: $\quad r > \emptyset$ unstable

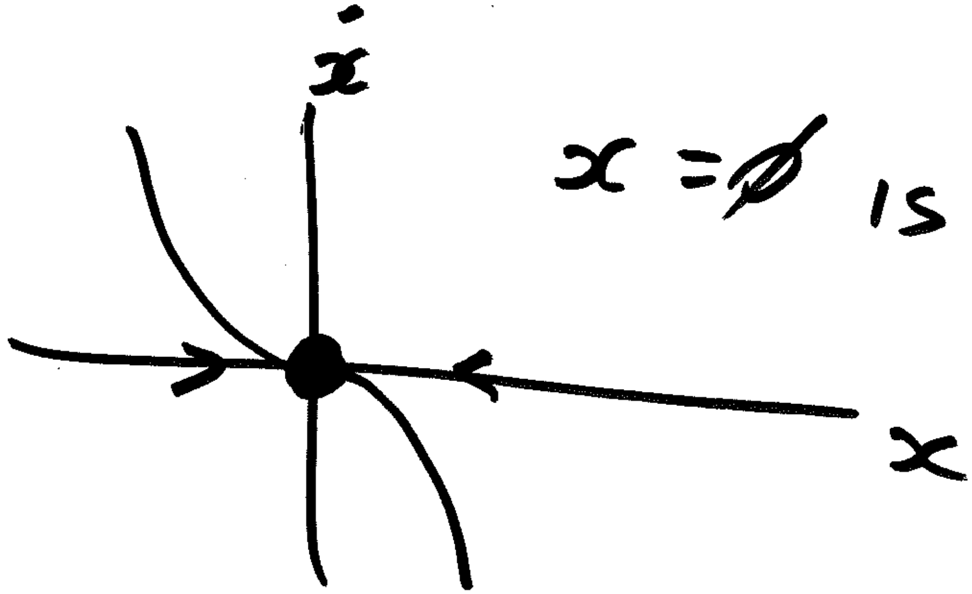
$r < \emptyset$ stable

$x^* = \pm\sqrt{r}$: $\underline{r > \emptyset}$:

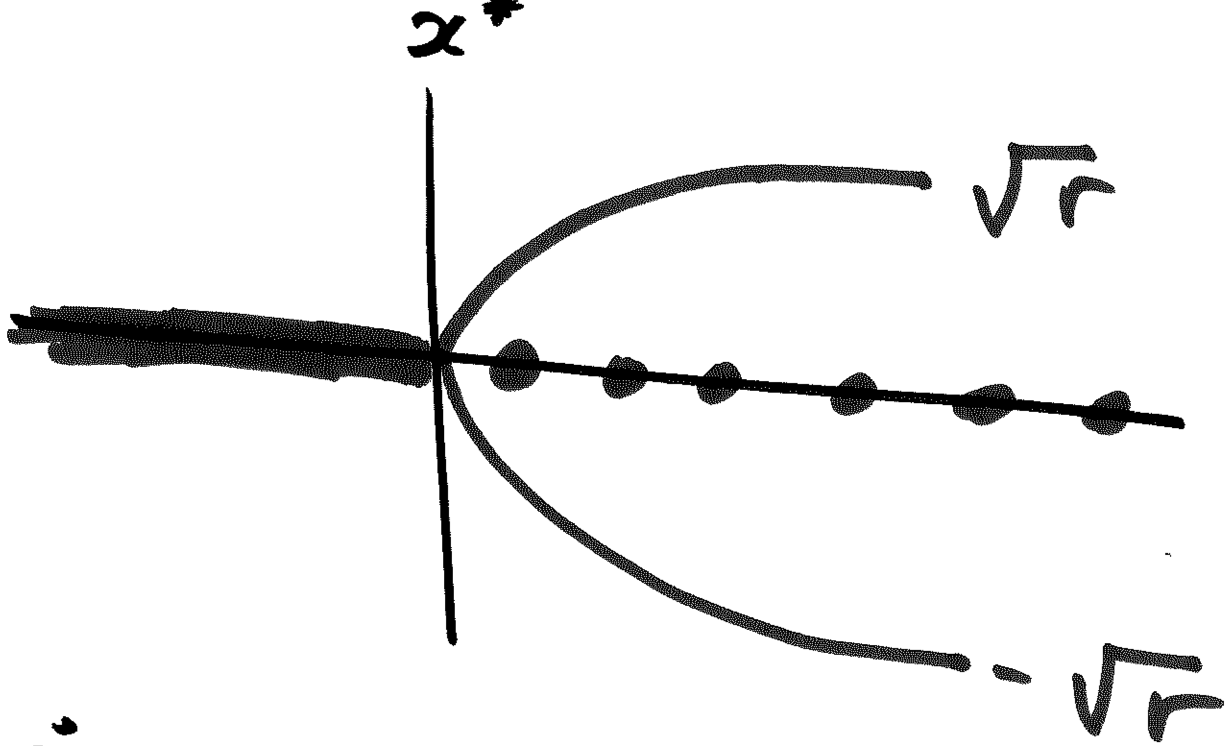
$\frac{\partial f}{\partial x} = r - 3(\pm\sqrt{r})^2 = -2r < \emptyset$

stable

$r = \emptyset$: $\dot{x} = -x^3$ 7

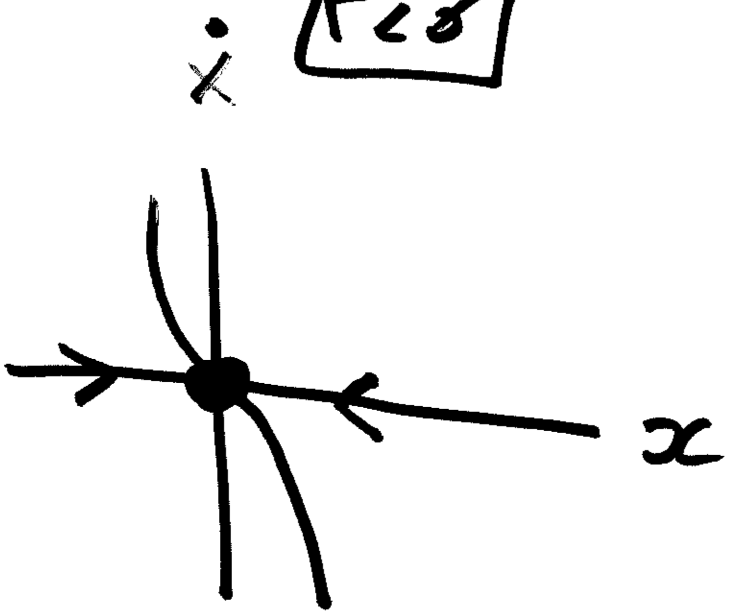


$x = \emptyset$ is stable

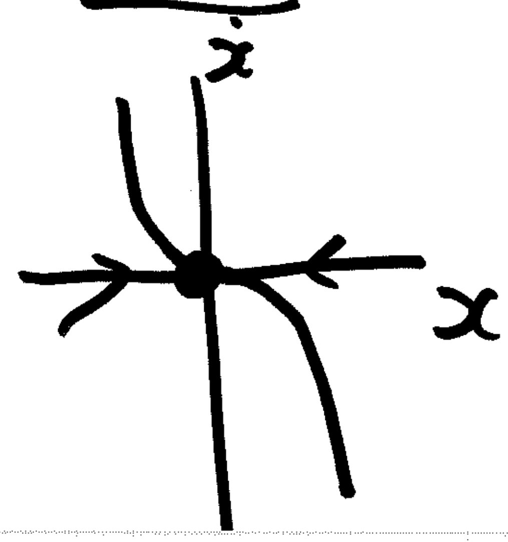


$$\dot{x} = rx - x^3$$

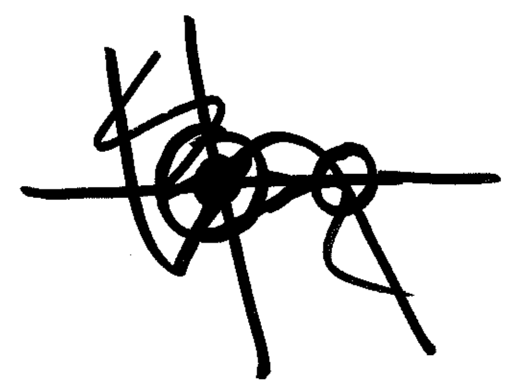
$\boxed{r<0}$



$\boxed{r=0}$



$r>0$



Consider $\dot{x} = -x + \beta \tanh x$;

Show that this system has supercritical pitchfork bifurcation, find β when bifurcation occurs.

Solution: solve $x = \beta \tanh x$;

$x = \emptyset$ is a solution;

Expand for small x:

$$\tanh x \approx x - x^3/3$$

small x: $\dot{x} = -x + \beta x - \beta x^3/3$

$$= x(\beta - 1 - \beta x^2/3)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$e^x = 1 + x + x^2/2 + x^3/6 \quad (e^x)' = e^x$$

$$\tanh x = \frac{1 + x + x^2/2 + x^3/6 - 1 + x - x^2/2 + x^3/6}{1 + x + x^2/2 + x^3/6 + 1 + x + x^2/2 + x^3/6}$$

$$= x$$

$$\dot{x} = x\left(\beta - 1 - \beta x^2/3\right)$$

$x = \emptyset,$

$$x = \pm\sqrt{\frac{3(\beta-1)}{\beta}}$$

$\beta = 1$

$$\dot{x} = x\left(r - x^2\right)$$

$x^* = \emptyset$ is stable for $\beta < 1$,
unstable for $\beta > 1$

$$\frac{dx(t)}{dt} = f(x, \beta)$$

$$f(x, \beta) = x\left(\beta - 1 - \beta x^2/3\right)$$

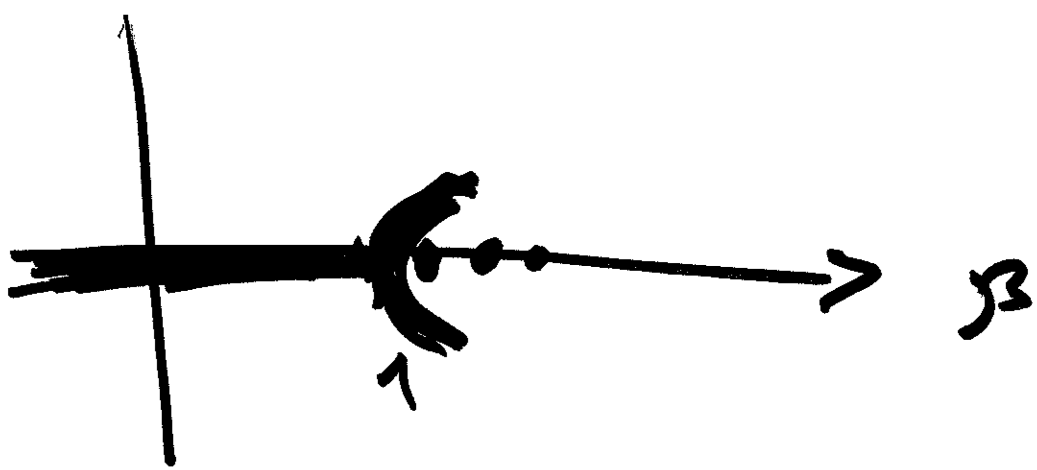
$$\frac{\partial f}{\partial x} = \beta - 1 - \beta x^2$$

<u>IF $x^* = \emptyset$</u>, $\frac{\partial f}{\partial x} = \beta - 1$, unstable $\beta > 1$
stable $\beta < 1$

IF $x^* = \pm\sqrt{3 - 3/\beta}$

$$\frac{\partial f}{\partial x} = \beta - 1 - \beta\left(3 - 3/\beta\right) =$$

$$= \beta - 1 - 3\beta + 3 = 2(1 - \beta)$$

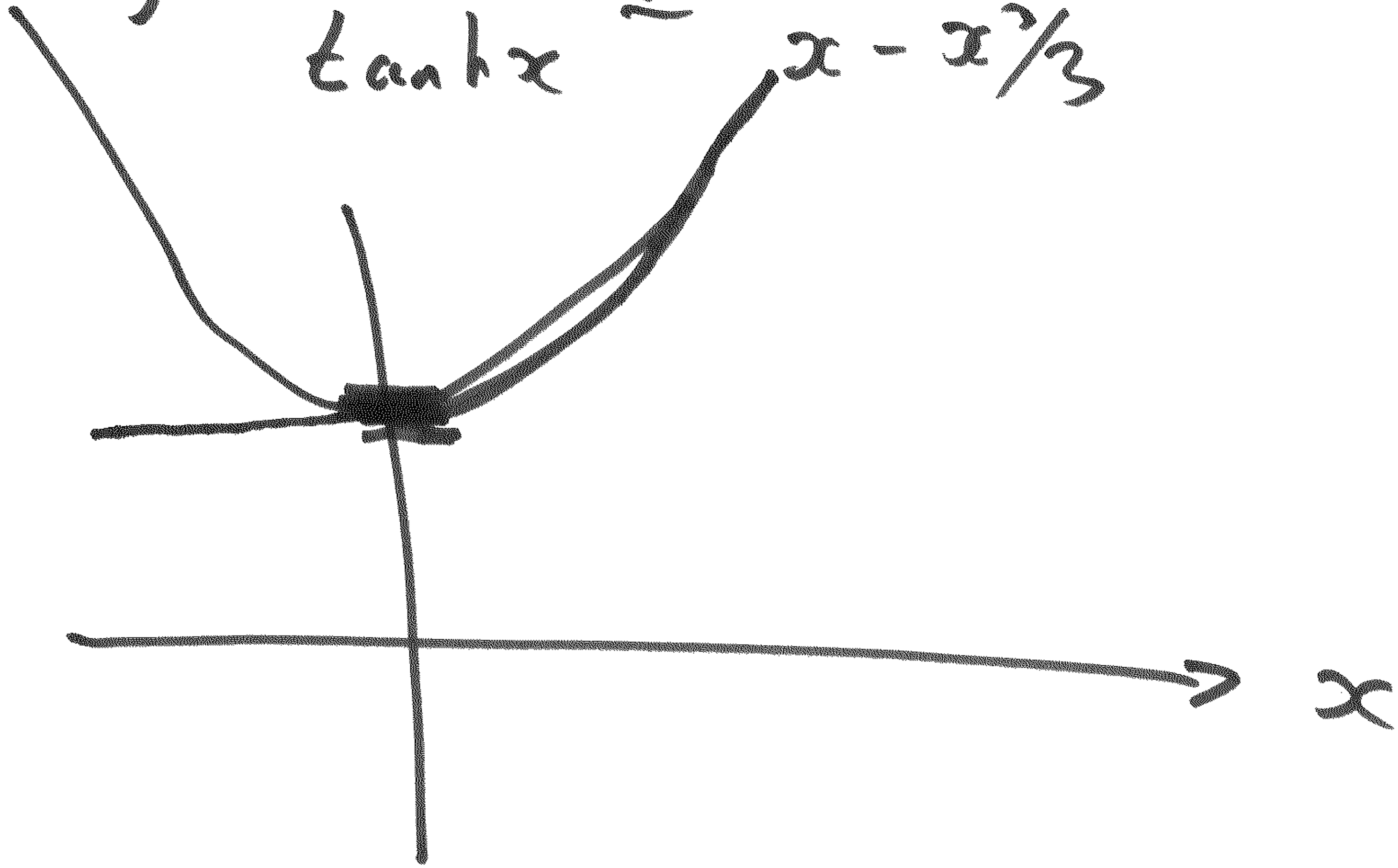


$$\dot{x} = -x + \beta \tanh x;$$

$$x = \beta \tanh x$$

$$\beta = \frac{x}{\tanh x}$$

$$\beta = \frac{x}{\tanh x} \approx \frac{x}{x - x^3/3}$$



Supercritical pitchfork: $\dot{x} = x(r - x^2)$

Subcritical pitchfork $\dot{x} = x(r + x^2)$

$\dot{x} = f(x, r) = x(r + x^2)$

$x^* = \emptyset$; $x^* = \pm\sqrt{-r}$

$$\frac{\partial f}{\partial x} = r + 3x^2$$

$x^* = \emptyset$: $\frac{\partial f}{\partial x} = r$

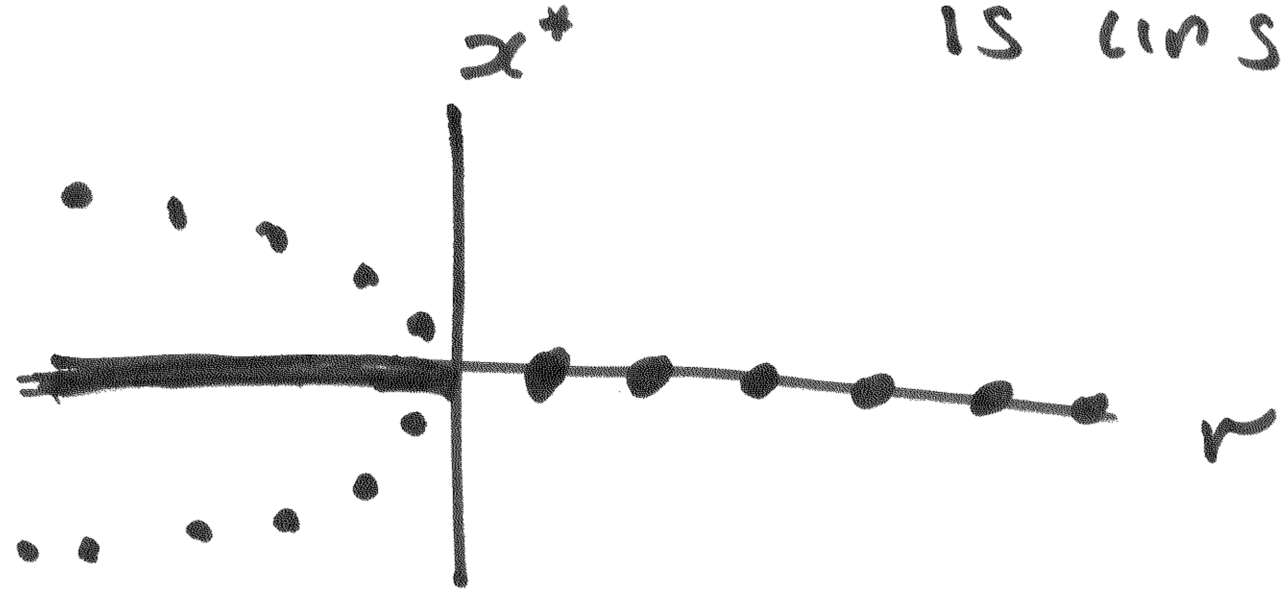
$r > \emptyset$ unstable

$r < \emptyset$ stable

$r = \emptyset$ unstable

$x^* = \pm\sqrt{-r}$; exists only $r < \emptyset$

$\frac{\partial f}{\partial x} = r + 3(-r) = -2r$, , is unstable



$$\dot{x} = x(r + x^2)$$

$$\int \frac{dx}{x(r+x^2)} = \int dt = t + C$$

$$\frac{1}{rx + x^3} = \frac{A}{x} + \frac{B + Cx}{r+x^2} = \frac{1}{rx} - \frac{1 \cdot x}{r(r+x^2)}$$

$$= \frac{Ar + Ax^2 + Bx + Cx^2}{x(r+x^2)} =$$

$$= \frac{Ar + Bx + x^2(A+C)}{x(r+x^2)}$$

$A = 1/r$

$B = \emptyset$

$C = -1/r$

$$\int \frac{dx}{x(r+x^2)} = \frac{1}{r}\left[\int \frac{dx}{x} - \int \frac{x\,dx}{r+x^2}\right]$$

$$= \frac{1}{r}\left[\ln x - \frac{1}{2}\ln(r+x^2)\right]$$

$$= t + C$$

$$x(t) = \pm \frac{\sqrt{r}}{\sqrt{A e^{-2rt} - 1}}$$

$$x(t=\emptyset) = \pm \frac{\sqrt{r}}{\sqrt{A-1}} = x_0$$

$$\dot{x} = x(r + x^2) - x^5$$

$\underline{x^* = \emptyset}$ fixed point

$$r + x^2 - x^4 = \emptyset \; ; \; z = x^2$$

$$r + z - z^2 = \emptyset$$

$$z^2 - z - r = \emptyset$$

$$z_{12} = \frac{1 \pm \sqrt{1 + 4r}}{2}$$

$$x^* = \pm \sqrt{\frac{1 \pm \sqrt{1 + 4r}}{2}}$$

$$x^* = \pm\sqrt{\frac{1 \pm \sqrt{1+4r}}{2}}$$

$1 + 4r > 0 \quad r > -1/4$

$\ominus \quad 1 - \sqrt{1+4r} > 0, \quad 1 > \sqrt{1+4r}, \quad r < 0$

$$x^*_{1,2} = \pm\frac{\sqrt{1-\sqrt{1+4r}}}{\sqrt{2}}; \quad -\frac{1}{4} < r < 0$$

$$x^*_{3,4} = \pm\frac{\sqrt{1+\sqrt{1+4r}}}{\sqrt{2}}; \quad r > -1/4$$

$$f(x,r) = x(r+x^2) - x5$$

$$\frac{\partial f}{\partial x} = r + 3x^2 - 5x^4$$

$x^* = \emptyset$: $r > \emptyset$ unstable

$r < \emptyset$ stable

$$x^*_{12} = \pm \frac{\sqrt{1-\sqrt{1+4r}}}{\sqrt{2}}$$

$$\frac{\partial f}{\partial x} = r + 3\left(\frac{1-\sqrt{1+4r}}{2}\right) - 5\left(\frac{1-\sqrt{1+4r}}{2}\right)^2$$

$$= r + \frac{3}{2} - \frac{3}{2}\sqrt{} - \frac{5}{4} - \frac{5}{4}(1+4r)$$

$$+ \frac{5}{4} \cdot 2\sqrt{}$$