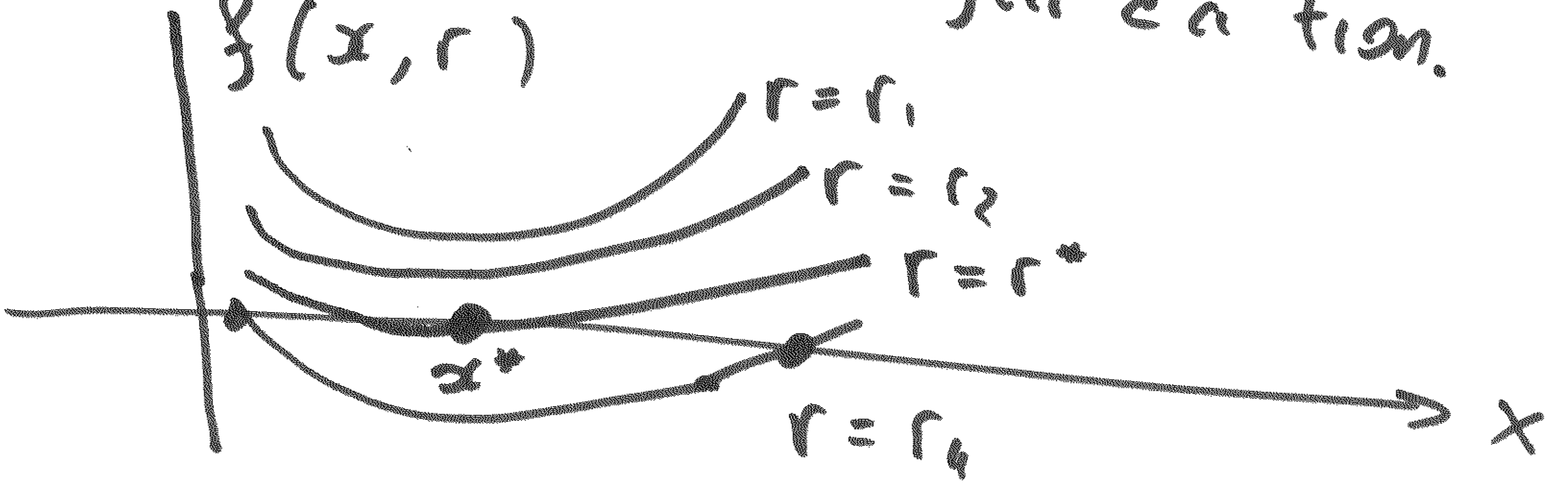
$\dot{x} = f(x, r)$

Saddle node bifurcation.



IF $r = r^*$ and $x = x^*$

$f(x, r^*)$ has extremum

$$\left.\frac{\partial f(x, r^*)}{\partial x}\right|_{x=x^*} = 0;$$

$$f(x^*, r^*) = 0$$

$$\frac{d}{dt} x(t) = f(x(t), r)$$

$$x(t) = x^* + \eta(t);$$

$$r = r^* + R$$

$$|R| << 1$$

$$|\eta| << 1$$

$$\frac{d}{dt}(\eta(t)) = f(x^* + \eta(t), r^* + R) =$$

$$= \underbrace{f(x^*, r^*)}_{=\emptyset} + \overbrace{\frac{\partial f(x^* r^*)}{\partial x}}^{=\emptyset} \eta + \frac{\partial^2 f(x^* r^*)}{\partial x^{*2}} \frac{\eta^2}{2}$$

$$+ \frac{\partial f(x^* r^*)}{\partial r} \cdot R$$

$$\frac{d}{dt}\,\eta(t) = \frac{\partial f(x^*, r^*)\,R}{\partial r} + \frac{\partial^2 f(x^*, r^*)}{2 \cdot \partial x^{*2}}\,\eta^2$$

$$\boxed{\dot{x} = r + x^2}$$

normal form

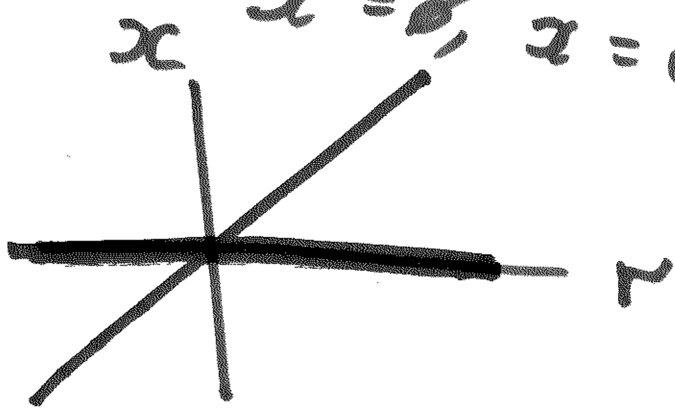
Blue sky bifurcation

Transcritical bifurcation.

$$\dot{x}(t) = rx - x^2$$

Fixed points: $\dot{x}(t) = f(x,r) = x(r-x) = \emptyset$

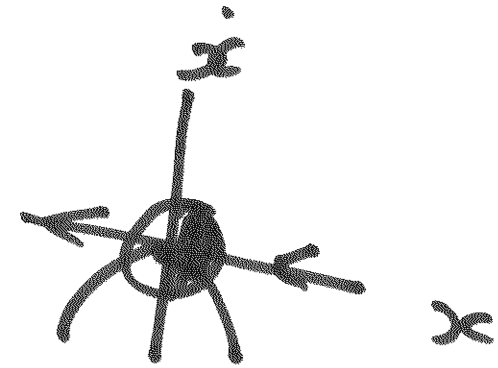
$x = \emptyset, \; x = r$



$$\frac{\partial f(x,r)}{\partial x} = r - 2x; \qquad x = \emptyset: \frac{\partial f}{\partial x} = r$$

$$\left[\begin{array}{l} r > \emptyset \text{ unstable} \\ r < \emptyset \text{ stable} \end{array}\right.$$

$r=\emptyset$ $\quad \dot{x} = -x^2$



$x=\emptyset$ is semistable fixed point

$x=r$: $\quad \dfrac{\partial f}{\partial x} = r - 2x = -r$; $\quad r>\emptyset$, $x=r$ is stable

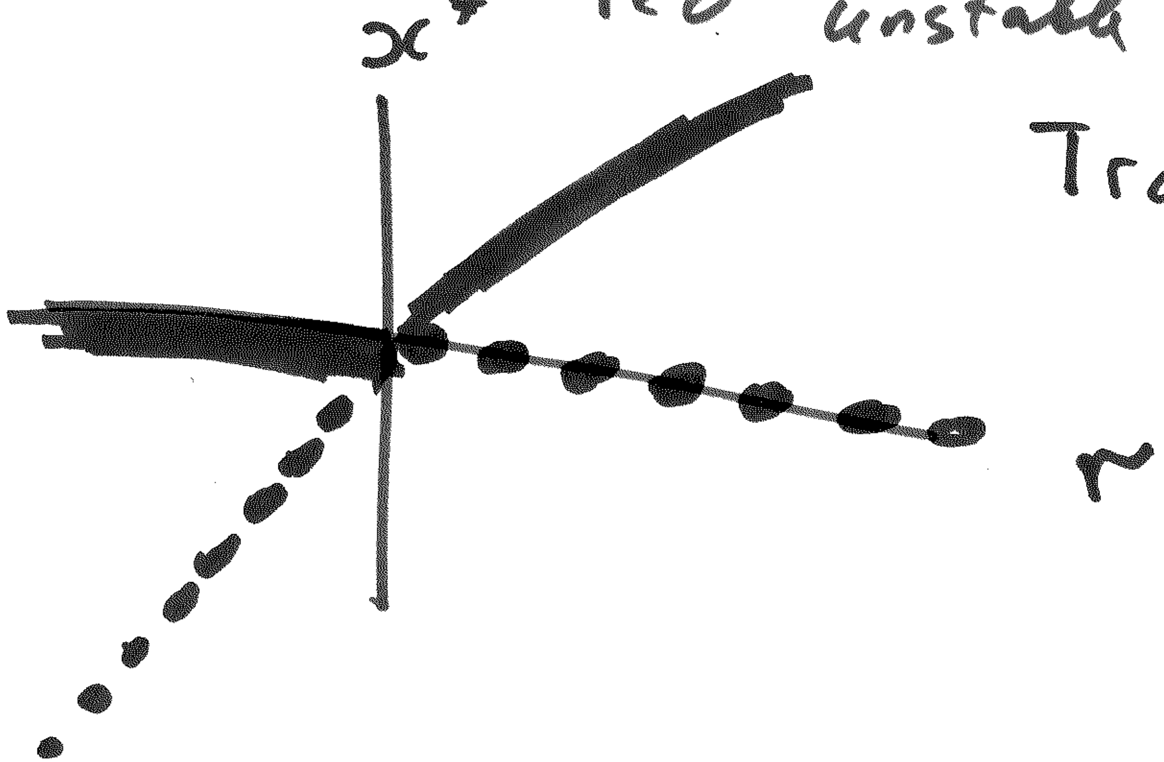
$r<\emptyset$ $\quad x=r$ is unstable

$x = 0$ $\quad r > 0$ unstable

$\quad\quad\quad\; r < 0$ stable

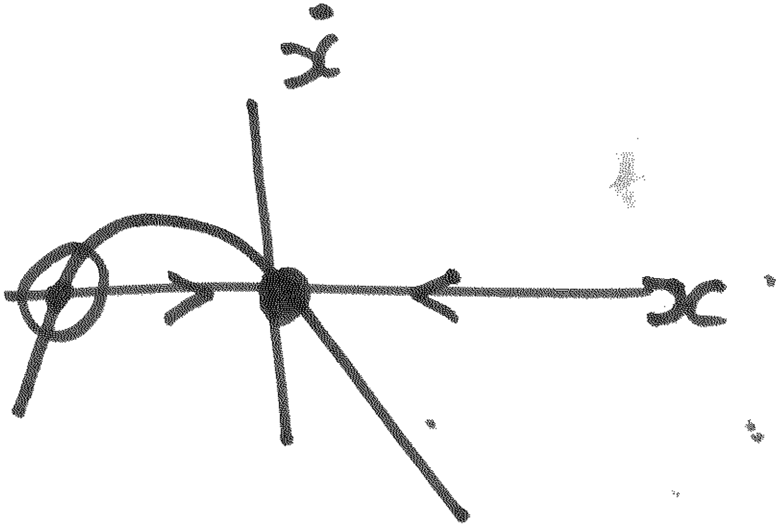
$x = r$ $\quad r > 0$ stable

$\quad\quad\quad\; r < 0$ unstable



Transc

$\dot{x} = rx - x^2$



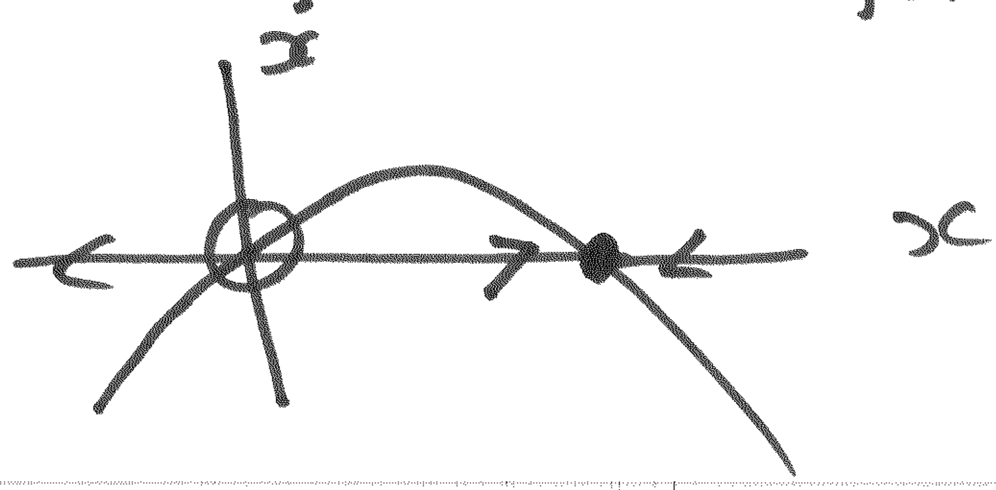
$r < 0$

$x = 0$ stable

$x = r$ is unstable

$r = 0$ both fixed points coincide and therefore mixed stability

$r > 0$



$x = r$ stable

$x = 0$ unstable

Example

Consider $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

Show it undergoes transcritical bifurcation, and find the curve in (a, b) space where bifurcation occurs.

Solution: $\dot{x} = f(x) = 0$,

$x(1-x^2) - a(1-e^{-bx}) = 0$

trancedental in x;

for any

$\downarrow$

$x = 0$ is a fixed point $\forall a, b$

Assume $|x| << 1$, expand small x:

$$e^{-bx} = 1 - bx + b^2x^2/2 - \dots$$

$$\dot{x} = x(1-x^2) - a(1 - e^{-bx}) \approx$$

$$\approx x(1-x^2) - a(1 - 1 + bx - b^2x^2/2)$$

$$= x(1-x^2) - abx + ab^2x^2/2$$

$$= x(1-ab) + ab^2x^2/2 - x^3 + \dots$$

$$= x(1 - ab + ab^2x/2) \cancel{-x^2} + \dots$$

$$\boxed{\dot{x} = x(r-x)} \qquad r = 1 - ab$$

Transcritical bifurcation occurs if $ab = 1$,

$$1 - ab + ab^2 x/2 = 0$$

$$\frac{ab^2 x}{2} = ab - 1$$

$$x = \frac{2(ab-1)}{ab^2} = \frac{2(ab-1)}{ab^2}$$

$$\dot{x} = x\left(1 - ab + ab^2 x/2\right)$$

$$x = \emptyset, \quad x = \frac{2(ab-1)}{ab^2}$$

$$\frac{\partial f}{\partial x} = 1 - ab + ab^2 x$$

- $x = \emptyset$:
  - $1 - ab > \emptyset$ unstable
  - $1 - ab < \emptyset$ stable
  - $1 = ab$ mixed stability
- $$\frac{\partial f}{\partial x} = 1 - ab + ab^2\left(\frac{2(ab-1)}{ab^2}\right)$$
  $$= 1 - ab + 2ab - 2$$
  $$= -(1 - ab)$$
  - $1 - ab < \emptyset$ unstable
  - $1 - ab > \emptyset$ stable

Example: $\dot{x} = r\ln x + x - 1 \quad (*)$

Investigate whether the bifurcation is transcritical, Change variables to bring $(*)$ to the normal form.

<u>Solution</u>

Find x^* such that

$$r\ln x^* + x^* - 1 = \cancel{0}\,;$$

By inspection, $x^* = 1$ is a fixed point.

$x = 1 + u$; $|u| << 1$; $\ln(1+u) \approx u - u^2/2$

$\cancel{\dot{x}}$

$$\dot{u} = r\left(u - \frac{u^2}{2}\right) + u = u\left(r + 1 - \frac{ru}{2}\right)$$

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$|x| << 1$

$f(x) = \ln(1+x)$.

$f'(x) = \dfrac{1}{1+x}$

$f''(x) = -\dfrac{1}{(1+x)^2}$

$$f(x) = f(\emptyset) + f'(\emptyset)x + f''(\emptyset)x^2/2$$
$$= \ln(1) + 1 \cdot x - \frac{1}{2}x^2$$
$$= x - x^2/2$$

(c)

$\dot{u} = u(r+1 - ru/2)$ ①

$u = \emptyset;$ $\quad \dfrac{2(r+1)}{r} = 2 + 2/r = u^*$ ②

$\dfrac{\partial f}{\partial u} = r+1 - ru;$ ③ $\quad \boxed{\begin{array}{l} r+1-ru/2=\emptyset \\ 2r+2 = ru \end{array}}$

Bifurcation occurs at $r = -1$ ④

$u(t) = \frac{2}{r} v(t);$

$\dot{u}(t) = \frac{2}{r} \dot{v}(t)$ $\quad \frac{2}{r}\dot{v}(t) = \frac{2}{r} v(t)\left(r+1-\frac{r}{2}\frac{2}{r}v\right)$

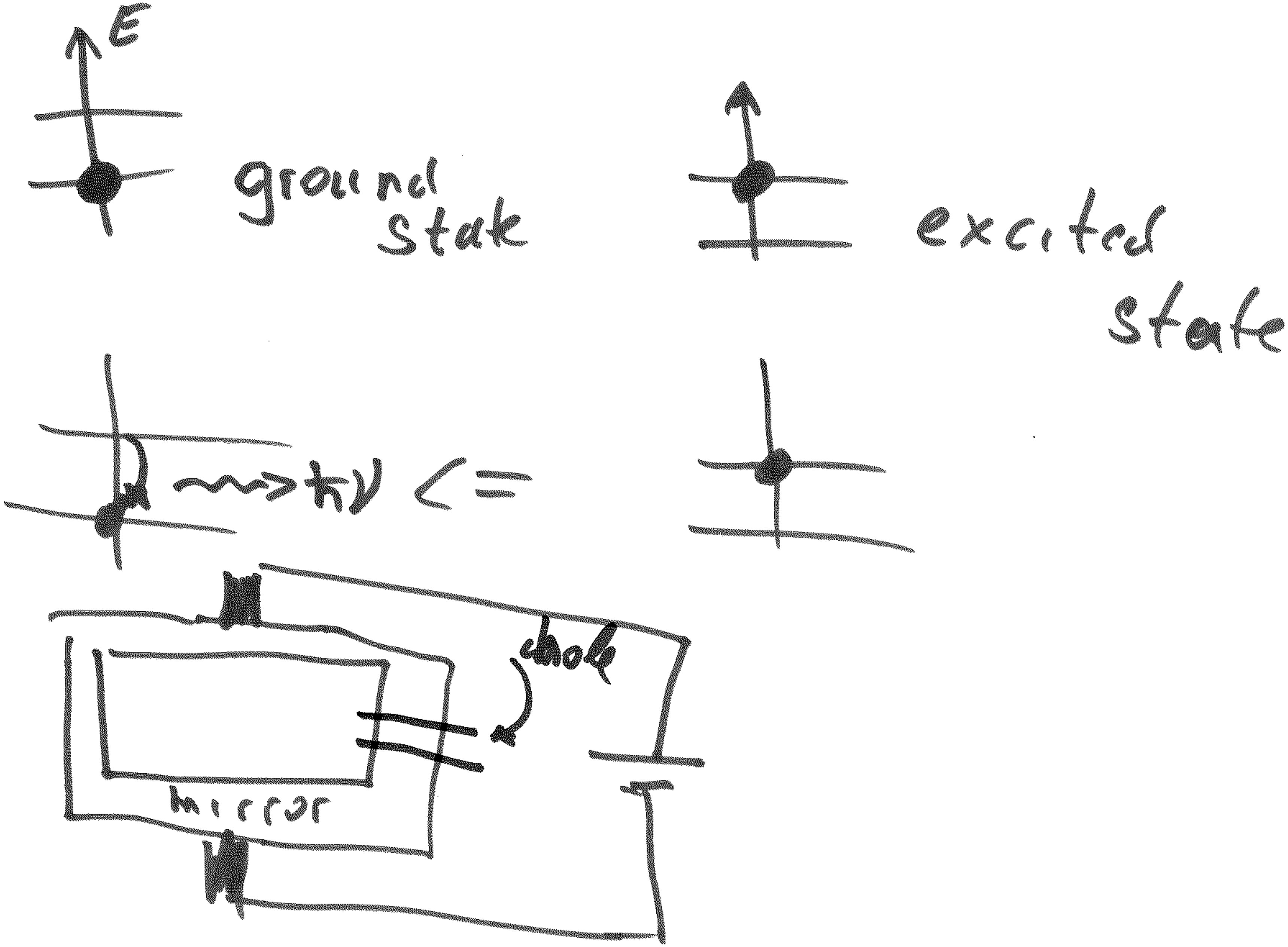
$\dot{v}(t) = v(t)(r+1-v)$

$r+1 = R$

Laser Threshold.

Light
Amplification
by
Spontaneous
Emission
of
Radiation
.

Active medium 1960



$n(t)$ is # of photons in a laser.

$$\frac{d}{dt} n(t) = \text{In} - \text{Out}$$

$$\text{In} = G\, n\, N$$

N → # of excited atoms

n → # of photons

gain coefficient

$G > \cancel{0}$

$$\text{Out} = k \cdot n$$

k → is inverse lifetime of a photon

$$N(t) = N_0 - \alpha\, n(t),$$

$$\frac{dn(t)}{dt} = GnN - kn$$

$$= Gn(N_0 - \alpha n) - kn$$

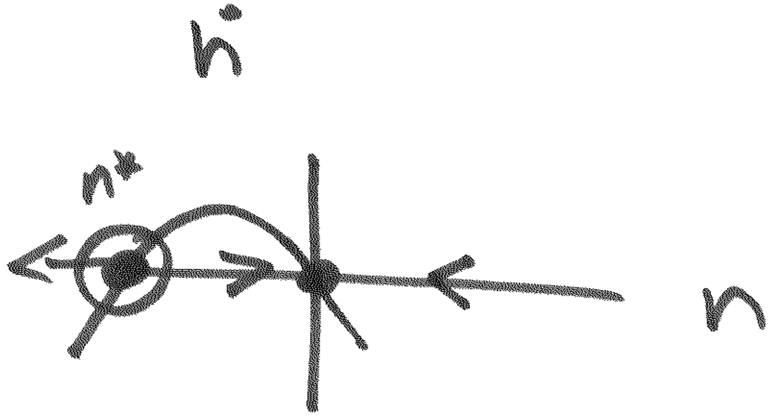
$$= n(GN_0 - k) - \alpha G n^2$$

Fixed points: $n = \emptyset$

$$\frac{GN_0 - k}{\alpha G} = n^*$$

$$\dot{n} = n(GN_0 - \beta - \alpha G n)$$

- $GN_0 - \beta < 0$

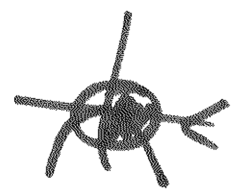


0 stable

$$n^* = \frac{GN_0 - \beta}{\alpha G} < 0$$

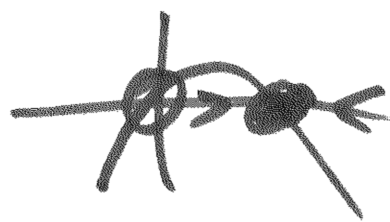
is unstable

- $GN_0 - \beta = 0$:



n^* is semi stable

- $GN_0 - \beta > 0$



$n^* > 0$ stable

$n = 0$ unstable