A

$\dot{x}(t) = f(x(t))$;

x^* such that $f(x^*) = 0$;

$x(t) = x^* + \eta(t)$

$$\frac{d}{dt}\eta(t) = f'(x^*)\eta(t) + f''(x^*)\eta^2(t)/2 + f'''(x^*)\eta^3/6 + \dots$$

- $f'(x^*) \neq 0$: $\eta(t) = \eta(t=0)\, e^{f'(x^*)t}$;

  $f'(x^*) > 0$, unstable,

  $f'(x^*) < 0$, stable fixed point.

- $f'(x^*) = 0$, $f''(x^*) \neq 0$: x^* is semistable;

$$\frac{d}{dt}\eta(t) = f''(x^*)/2 \cdot \eta^2(t); \quad a \equiv f''(x^*)/2;$$

B

$$\frac{d}{dt}\eta(t) = a\eta^2(t);$$

$$\int \frac{d\eta(t)}{\eta^2} = \int a\,dt = at + C = -\frac{1}{\eta(t)}$$

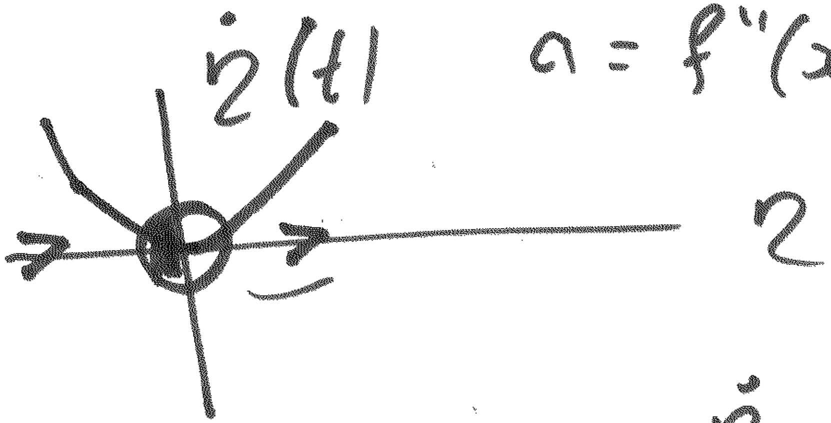
$$\eta(t) = -\frac{1}{C+at} = \frac{1}{\tilde{C}-at}$$

$$\eta(t) \text{ at } t=0 \Rightarrow \eta(t=0) = \frac{1}{\tilde{C}} = \eta_0,$$

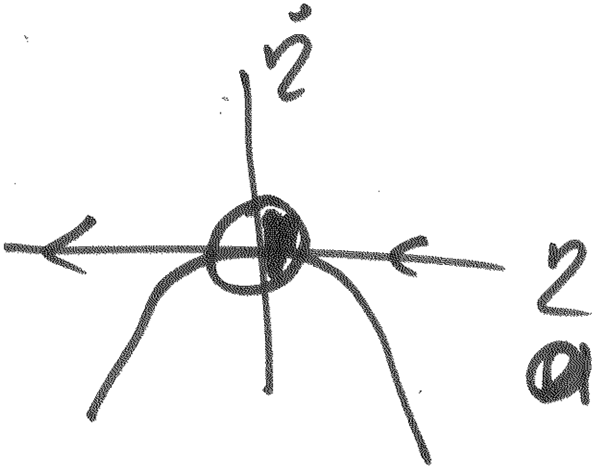
$$\tilde{C} = 1/\eta_0$$

$$\eta(t) = \frac{1}{1/\eta_0 - at} =$$

$$= \frac{\eta_0}{1 - a\eta_0 t}$$



$a = f''(x^*)/2 > 0$



$a = f''(x^*)/2 < 0$

$$\eta(t) = \frac{\eta_0}{1 - a\eta_0 t}$$

$a > 0$, $\eta_0 < 0$: $\eta(t) = \dfrac{\eta_0}{1 + |a\eta_0 t|}$

$\eta_0 > 0$: $\eta(t) = \dfrac{\eta_0}{1 - |a\eta_0 t|}$

D

$$f(x^*) = f'(x^*) = f''(x^*) = 0$$

$$\frac{d}{dt}\eta(t) = f'''(x^*)/6 \quad \eta^3(t);$$

$$b \equiv f'''(x^*)/6$$

$$\dot{\eta}(t) = b\,\eta^3(t) \qquad \eta(t) = \frac{1}{\sqrt{\frac{1}{\eta_0^2} - 2bt}}$$

$$\int \frac{d\eta}{\eta^3} = \frac{\eta^{-2}(t)}{-2} = \int b\,dt = bt + c$$

$$\eta^{-2}(t) = -2bt + \tilde{c}$$

$$\eta^{-1}(t) = \sqrt{-2bt + \tilde{c}}\,; \quad \eta(t) = \frac{1}{\sqrt{\tilde{c} - 2bt}}$$

$$\eta^{-2}(t) = \tilde{C} - 2bt$$

$$\eta^{2}(t) = \frac{1}{\tilde{C} - 2bt}$$

$$\eta(t) = \pm\frac{1}{\sqrt{\frac{1}{\eta_0^2} - 2bt}}$$

$$\eta(t) = \frac{\pm\,\eta_0}{\sqrt{1 - 2bt\,\eta_0^2}}$$

IF $b \equiv f'''(x^*)/6 < 0$

·k

IF $b \equiv f'''(x^*)/6 < 0$, $\lim_{t \to \infty} z(t) = 0$

stable

IF $b \equiv f'''(x^*)/6 > 0$, $\lim z(t) = \pm\infty$

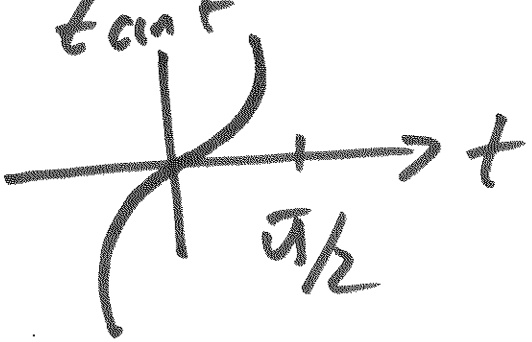
Example:

$$\dot{x}(t) = 1 + x^2(t)$$
$$x(t=0) = 0;$$

"blow up":

$$\frac{dx(t)}{1+x^2(t)} = \int dt = t + C = \text{Arc Tan } x(t)$$

$$x(t) = \tan(t + C)$$
$$x(t=0) = 0 \Rightarrow C = 0 : \quad x(t) = \tan(t)$$



$$\lim_{t \to \pi/2 - 0} x(t) = \infty$$

Potential:

$$\dot{x}(t) = f(x(t)) = -\frac{d}{dx} \overset{\text{potential}}{V}(x(t))$$

$$f(x) = -\frac{d}{dx} V(x)$$

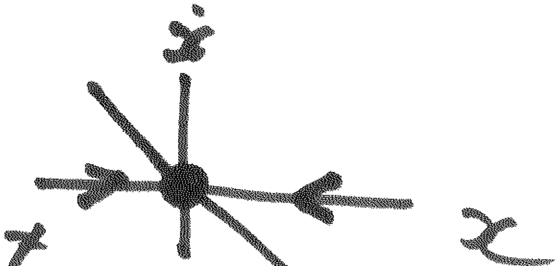
$$V(x) = -\int dx\, f(x);$$

$$\frac{d}{dt} V(x(t)) = \frac{dV(x(t))}{dx} \frac{dx(t)}{dt}$$

$$= -\left(\frac{d}{dx} V(x)\right)^2 \leq 0$$

I

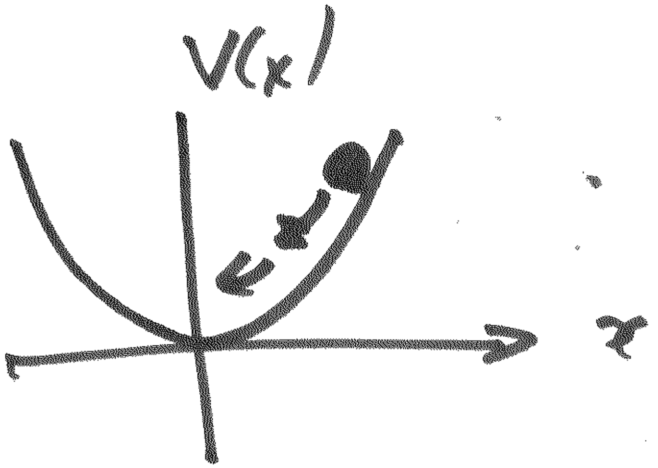
$\dot{x} = -x;$



$x(t) = x_0 e^{-t};$

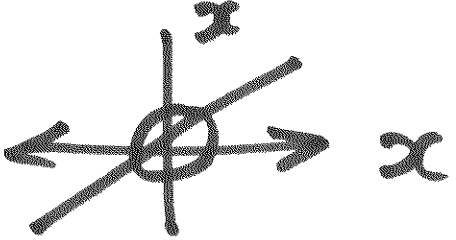
$$v(x) = -\int f(x)dx = -\int(-x)dx = \frac{x^2}{2} + C$$

Potential is defined up to a constant.

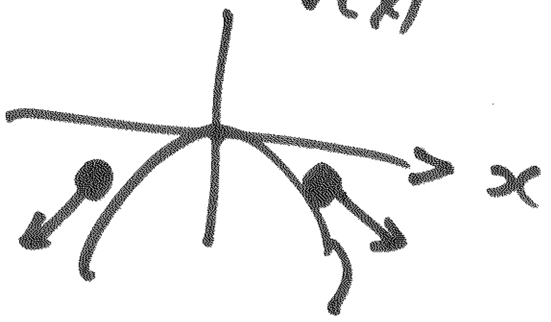


$\dot{x} = x$;

$x(t) = x_0 e^{t}$; unstable



$V(x) = - x^2/2$

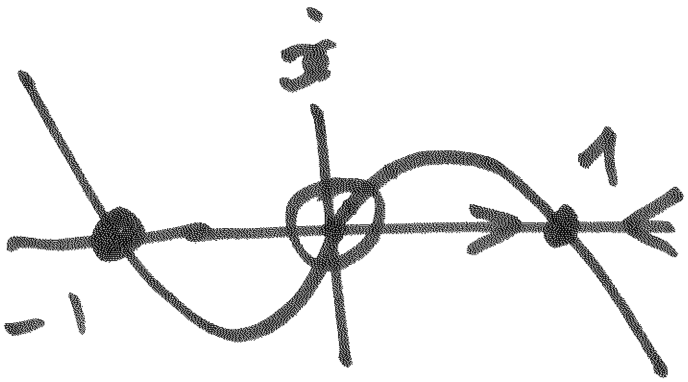


Example

K

$\dot{x} = x - x^3 \Rightarrow x^* = 0$ - unstable

$x^* = \pm 1$ - stable



$f(x) = x - x^3$

$f'(x) = 1 - 3x^2$

$x^* = 0 : f(x^*) = 1 > 0$ unstable

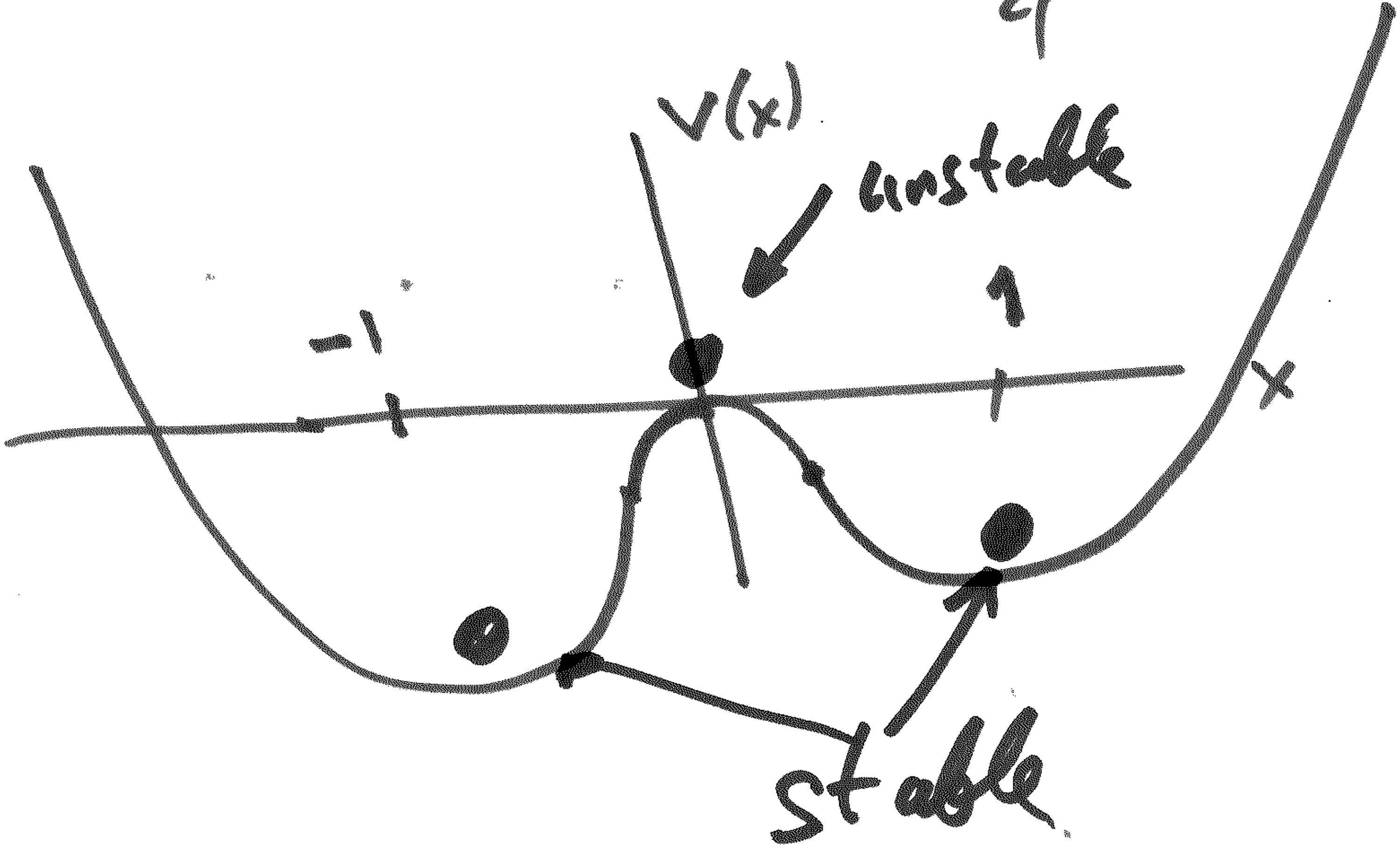
$x^* = 1 : f(x^*) = 1 - 3 = -2 < 0$ stable

Potential:

$$V(x) = -\int f(x)dx = -\int (x - x^3)\,dx = -\frac{x^2}{2} + \frac{x^4}{4}$$

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$$

L



M

$m\ddot{x} - b\dot{x} + kx = 0$

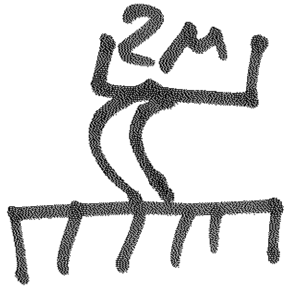
$\cdot \lim m \to 0 ; \; \cancel{m\ddot{x}} - b\dot{x} + kx = 0$

$\therefore \lim b \to \infty$

Bif~~f~~urcation.

N

Saddle Point Biffurcation



or

$\dot{x} = r + x^2$, r is external tunable parameter

Find fixed points:

$$\dot{x} = f(x, r) = r + x^2$$

$$f(x^*, r) = 0$$

$$x_*^2 = -r; \quad x^* = \pm\sqrt{-r}$$

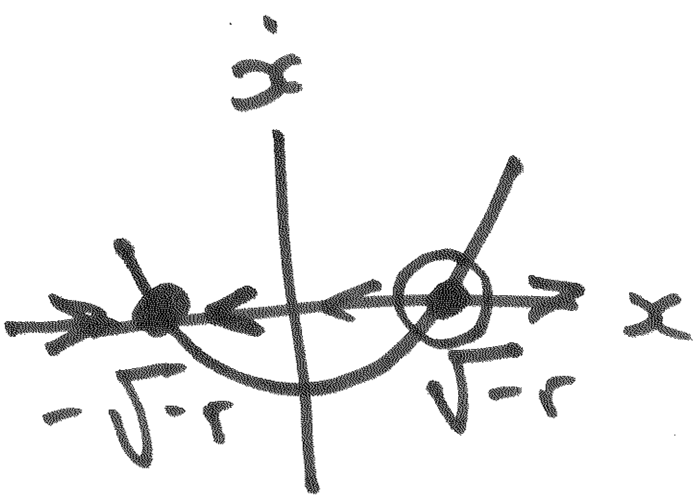
$x^* = \pm\sqrt{-r}$; $\dot{x} = r + x^2 = f(x,r)$ 0

$\underline{r < 0}$

$x_1^* = \sqrt{-r}$, $x_2^* = -\sqrt{-r}$

$\dfrac{\partial f(x,r)}{\partial x} = 2x$; $\dfrac{\partial f(x_1^*,r)}{\partial x} = 2\sqrt{-r} > 0$

unstable

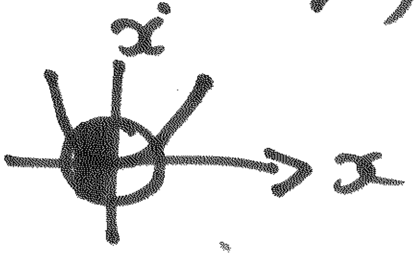


$\dfrac{\partial f(x_2^* r)}{\partial x} = -2\sqrt{-r} < 0$

stable

p

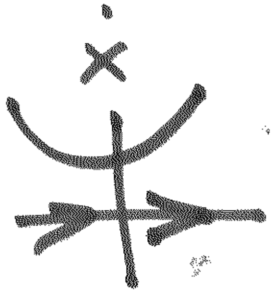
r = 0 $\dot{x} = x^2$; $x^* = 0$;



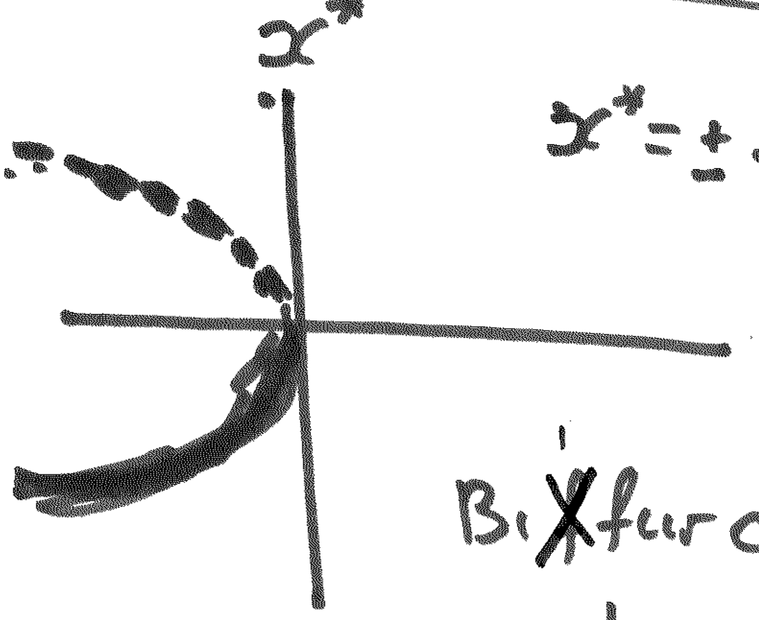
$x^* = 0$ is stable if $x = 0^-$ ← approaching 0 from left

unstable if $x = 0^+$ ← -||- right

r > 0



No fixed points



$x^* = \pm\sqrt{-r}$

Bifurcation diagram

Example:

Find bifurcation diagram for

$$\dot{x} = r - x - e^{-x}$$

and it is a saddle point bifurcation.

$$\dot{x} = f(x,r) = r - x - e^{-x}$$

x^* such that $f(x^*, r) = r - x_* - e^{-x_*} = 0$

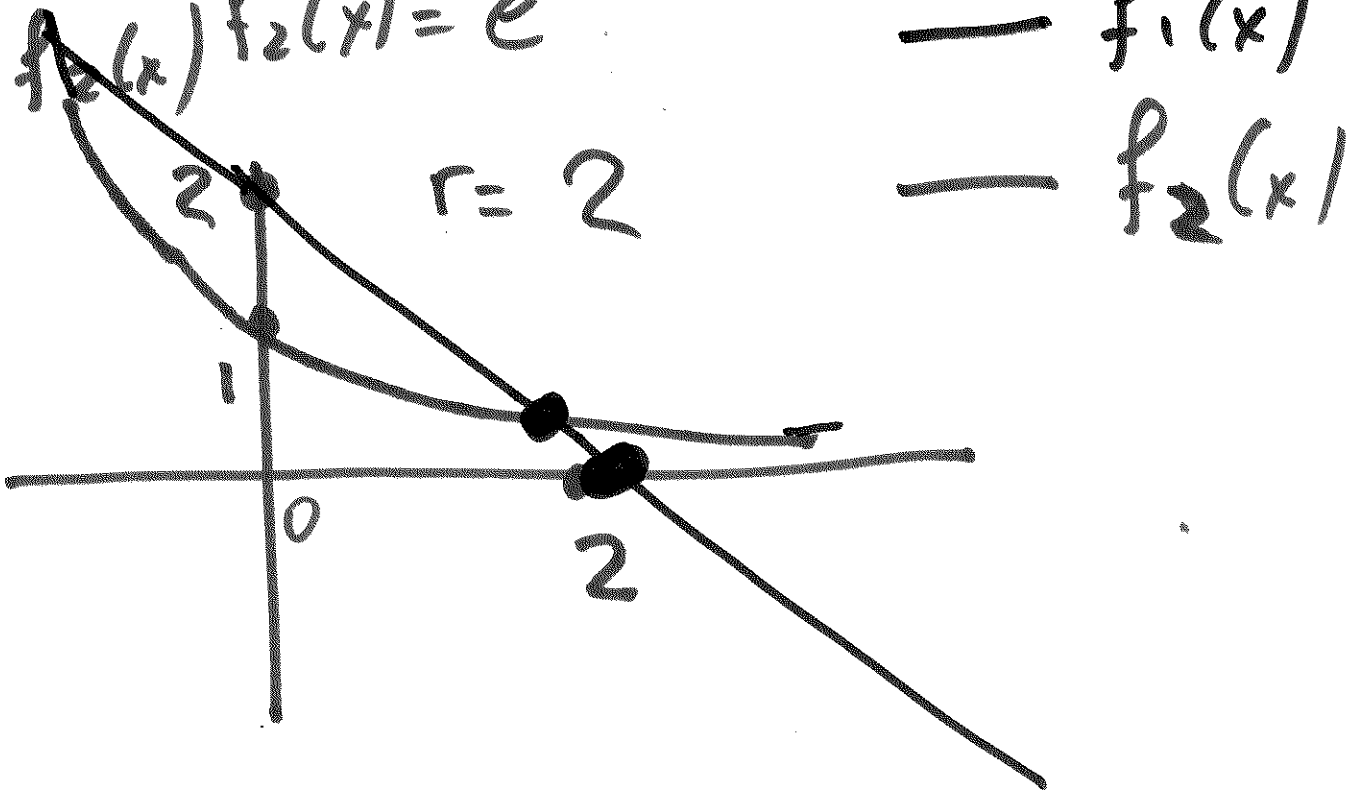
trancedental equation for x^*

R

$\dot{x} = f_1(x) - f_2(x);$

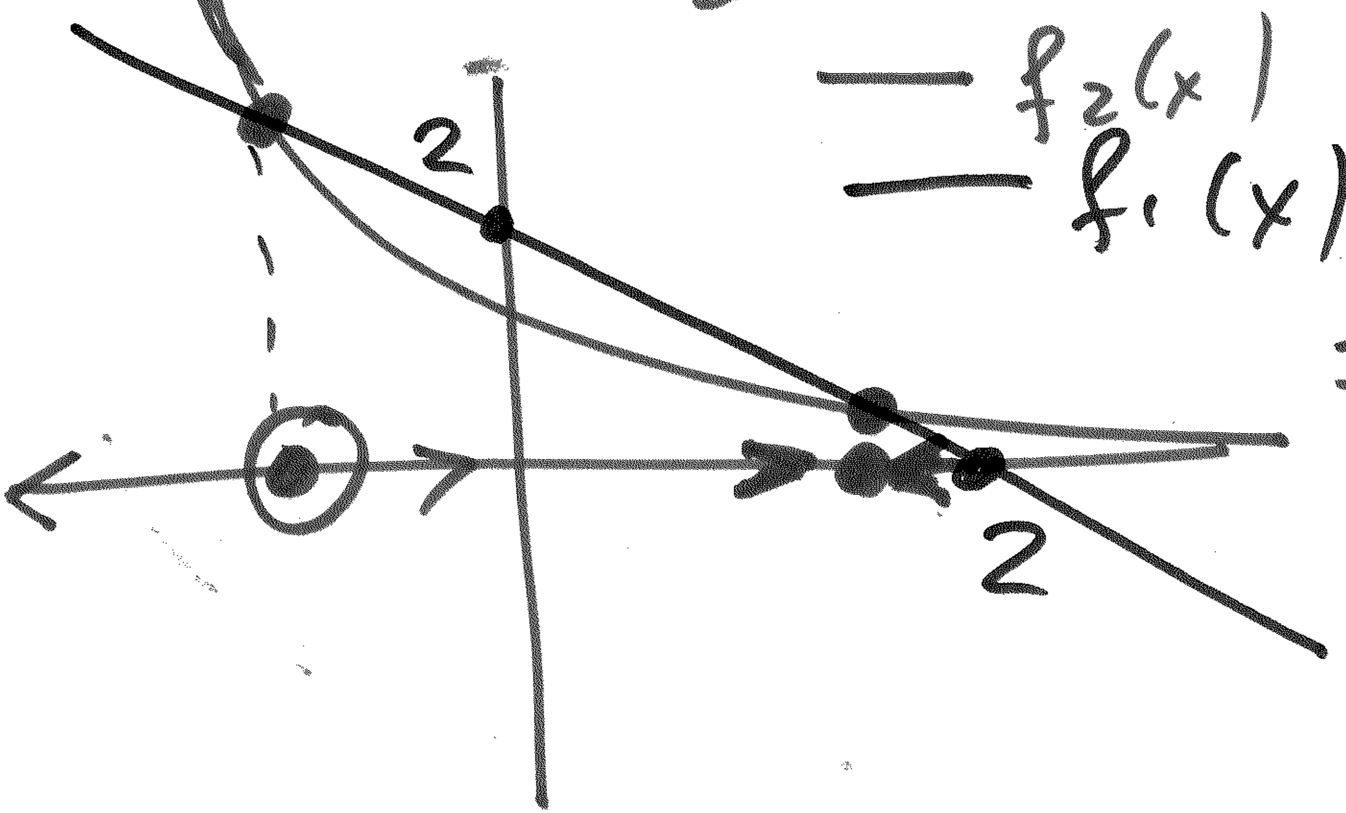
$f_1(x) = r - x$

$f_2(x) = e^{-x}$



$r = 2$

$x = r - x - e \quad f_1(x) - f_2(x)$ S



$\boxed{r=2}$

$$\dot{x} = r - x - e^{-x}$$
$$= f_1(x) - f_2(x)$$

Left one is unstable

Right one is stable

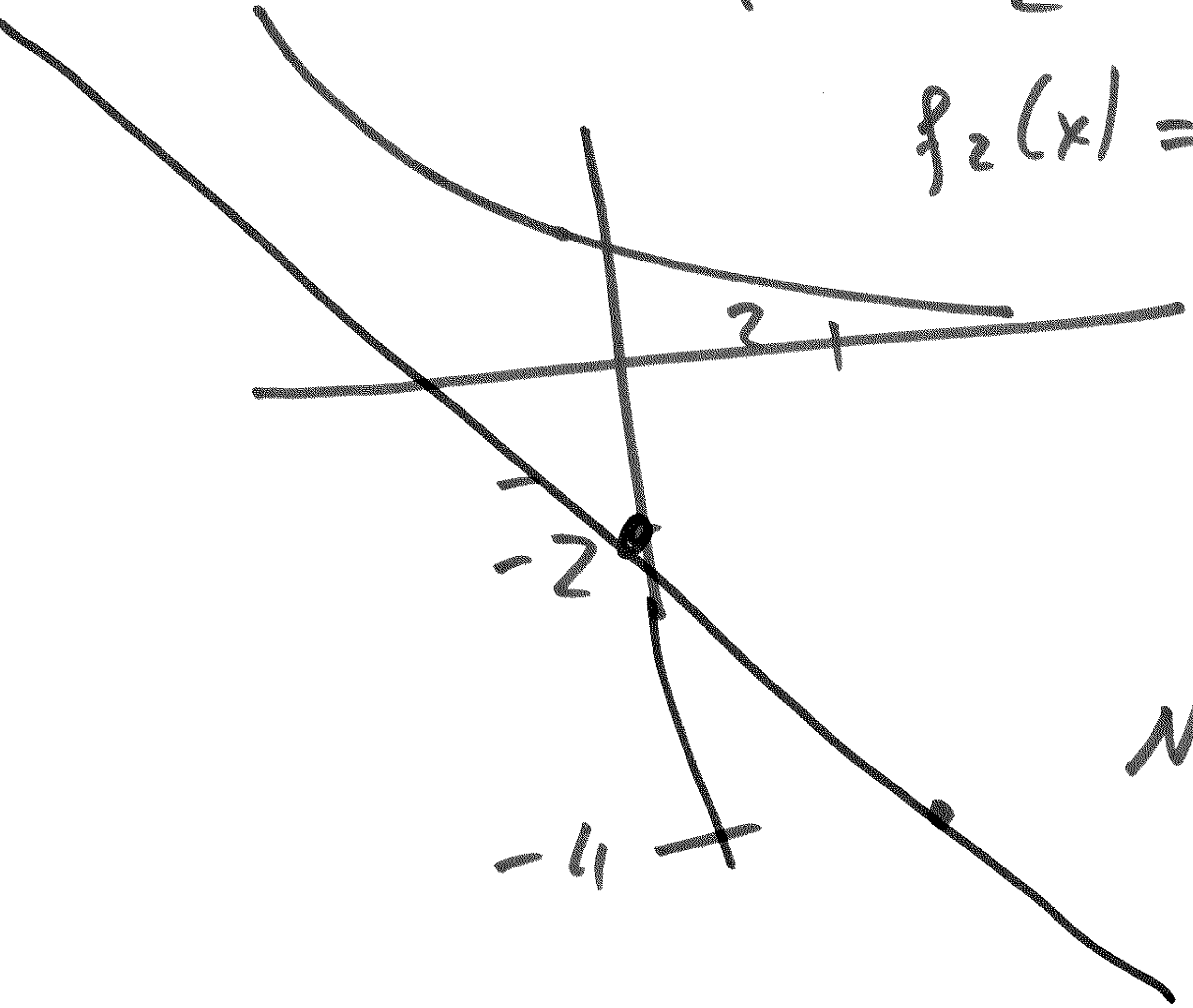
T

$r = -2$

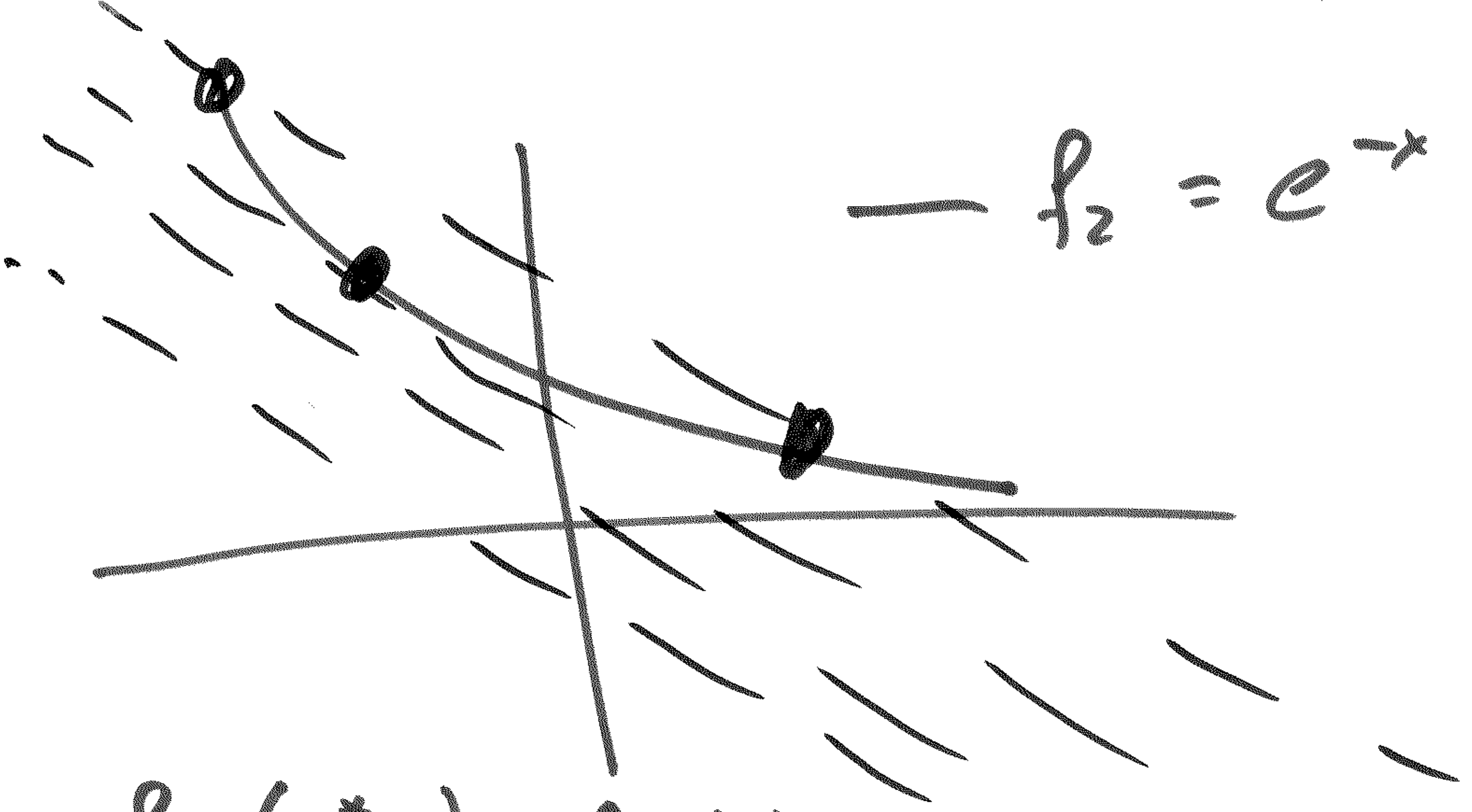
$f_2(x) = e^{-x}$

$\dot{x} = f_1(x) - f_2(x)$

$f_1 = r - x$



No fixed points



$$f_1(x^*, r) = f_2(x^*, r)$$

$$f_1(x, r) = r - x$$
$$f_2(x, r) = e^{-x}$$

$$\frac{\partial f_1(x^*, r)}{\partial x} = \frac{\partial f_2(x^*, r)}{\partial x}$$

$$f_1(0, r) = r$$
$$f_2(0, r) = 1$$

$$\Rightarrow \boxed{r = 1}$$

$$-1 = -e^{-x^*}$$
$$1 = e^{-x^*} \Rightarrow x^* = 0$$

Bifurcation $x = 0, \ r = 1$

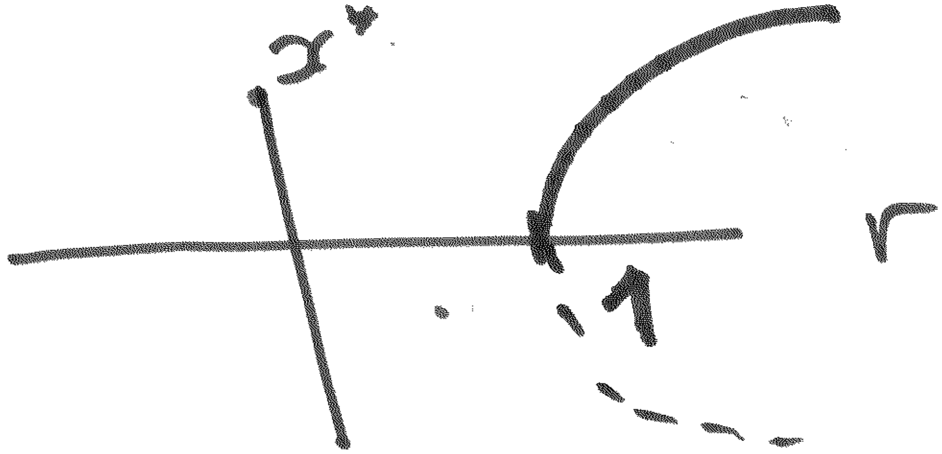
$r > 1$, two fixed points

Left - unstable

Right - stable

$r = 1$: bifurcation occurs

$r < 1$ - no fixed points



$r - x - e^{-x} = 0$

$r = x + e^{-x}$

Bifurcation occurs at ✓

$r^* = 1, \; x^* = 0;$

$\dot{x} = r - x - e^{-x}$ $\qquad e^{-x} \approx 1 - x + x^2/2$

$r = 1 + R$

$x = 0 + \tilde{x}, \quad |x| << 1$

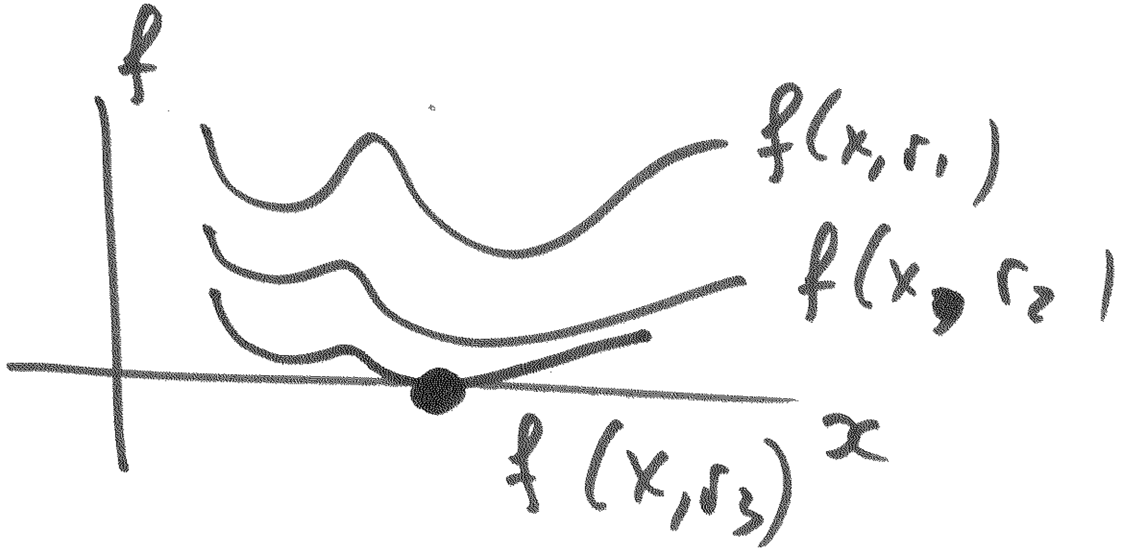
~~$\dot{x} = 1+R-(1-x$~~

$$\dot{x} = 1 + R - x - (1 - x + x^2/2 - \dots)$$

$$= R - x^2/2 \Rightarrow R = -\tilde{R}$$

$$x = -\tilde{x}$$

$\dot{x} = f(x, r)$ ✓



For ~~Bifurf~~ Bifurcation to occur,

$$\frac{\partial f(x^*, r)}{\partial x} = 0$$

W

$$\dot{x} = f(x, r)$$

$$x = x^* + \eta(t)$$

$$r = r^* + \rho$$