$\ddot{x} + (\dot{x})^3 + x = 0$ - nonlinear oscillator

Show, there is no periodic orbit in this system.

$$E(t) = \frac{x^2(t) + \dot{x}^2(t)}{2}$$ "energy"

$$\frac{d}{dt} E(t) = x(t)\dot{x}(t) + \dot{x}(t)\ddot{x}(t)$$
$$= \dot{x}(t)\left(x(t) + \ddot{x}(t)\right) =$$
$$= -\dot{x}(t) \cdot \left(\dot{x}(t)\right)^3 = -\left(\dot{x}(t)\right)^4 \leq 0$$

Assume there is a periodic orbit:

$$x(t+T) = x(t)$$
$$\dot{x}(t+T) = x(t)$$

Change of energy over periodic orbit:

$$\Delta E = E(t+T) - E(t)$$

equal

$$= \frac{1}{2}\left(x^2(t+T) + \dot{x}^2(t+T) - x^2(t) - \dot{x}^2(t) \right)$$

equal

$$= 0$$

$$\Delta E = \int_{t}^{t+T} \dot{E}(\tau)\,d\tau = E(t+T) - E(t) = 0$$

$$\rightarrow = -\int_{t}^{t+T} \left(\dot{x}(\tau)\right)^4 d\tau \leq 0$$

IF there is "movement" then $\dot{x}(\tau) \neq 0$, $\Delta E < 0$

contradiction

equal: $\dot{x}(\tau) = 0$

no movement

Lapunov function 1857 - 1918
Russia

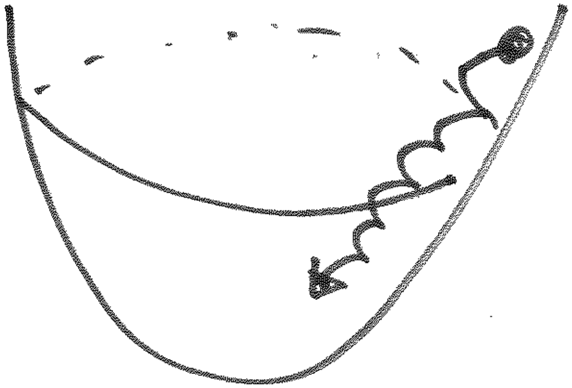
Energy-like function

$V(x)$ with properties

① $V(\underline{x}) > 0$ for any $\underline{x} \neq \underline{x}_0$

with $V(\underline{x}_0) = 0$

② $\dot{V}(\underline{x}) < 0$: Lapunov function decreases.



Example: $\dot{x} = -x + 4y$
$\dot{y} = -x - y^3$

Show, there is no closed orbits.

This is NOT gradient

Detour: $\dot{x} = \frac{\partial V}{\partial x}$

$\dot{y} = \frac{\partial V}{\partial y}$ $\Rightarrow$ $\frac{\partial}{\partial y}\dot{x} - \frac{\partial}{\partial x}\dot{y} = 0$

$$\frac{\partial}{\partial y}\dot{x} = \frac{\partial^2 V}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\dot{y} = \frac{\partial^2 V}{\partial x \partial y}$$

end of detour

$$\frac{\partial}{\partial y}\dot{x} = 4 \neq \frac{\partial}{\partial x}\dot{y} = -1$$

Find Lapunov

$\dot{x} = -x + 4y$

$\dot{y} = -x - y^3$

$V(x, y) = x^2 + y^2 \cdot a$ Not going to work

$$\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} =$$

$$= 2x\dot{x} + 2y\dot{y} \cdot a$$

$$= 2x(-x + 4y) + 2y(-x - y^3) a$$

$$= -2x^2 + 8xy - 2xya - 2y^4 \cdot a$$

$V(x,y) = x^2 + 4y^2$

$\dot{V}(x,y) < 0$

$V(x=0, y=0) = 0$

$8xy - 2axy = 0$

$8xy = 2axy$

$8 = 2a \Rightarrow a = 4$

Dulac Criterion, Henri Dulac

France
1870 - 1955

Consider $\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t))$ with $\underline{f}(\underline{x})$ being continiously differentiable in R.

Let there be continously differentiable $g(\underline{x})$ in R such that. $\nabla\cdot(g(\underline{x})\cdot\underline{\dot{x}})$ has one sign. Then there is no closed orbits in this system in R.

Suggestions: $g(x,y) = \dfrac{1}{xy}$; $g(x,y) = x^{\alpha} y^{\beta}$

$g(x,y) = e^{-x}, e^{-y}, e^{-xy}, e^{-x^2y^2}$ ....

Demonstrate: Let C be a closed orbit, $C \in R$.

Let there be $g(\underline{x})$ s.that $\nabla\cdot(g\dot{\underline{x}})$ be sign definite

$$\iint\limits_{\substack{\text{Area}\\ \text{in } C}} \nabla\cdot(g\dot{\underline{x}})\,dA$$ has one sign.

Green's theorem

$$0 < \iint\limits_{\substack{\text{Area}\\ \text{in } C}} \nabla\cdot(g\dot{x})\,dA = \oint_C g\,\underbrace{\dot{\underline{x}}\cdot\overset{\substack{\text{normal}\\ \downarrow}}{\underline{n}}}_{\text{zero}}\,\overset{\substack{\text{element of } C\\ \downarrow}}{dl} = 0$$

Example

$$\dot{x} = x(2-x-y);$$
$$\dot{y} = y(4x-x^2-3)$$

Show, there is no closed orbits for $x>0, y>0$.

$$g(x,y) = \frac{1}{xy}$$

• no closed orbits

$$\nabla\cdot\left(g(\underline{x})\cdot\dot{\underline{x}}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left[g(x)\begin{pmatrix} x(2-x-y) \\ y(4x-x^2-3)\end{pmatrix}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{xy}x(2-x-y)\right) + \frac{\partial}{\partial y}\left[\frac{1}{xy}y(4x-x^2-3)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x}(4x-x^2-3)\right) = -1/y < 0$$

Example

$$\dot{x} = x^2(2-x-y)$$

$$\dot{y} = y^2(4x-x^2-\sin x)$$

Show, there is no closed orbits for $x>0$, $y>0$

$$g(x,y) = \frac{1}{x^2y^2}$$

$$\nabla\cdot\left(g(\underline{x})\cdot\dot{\underline{x}}\right) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\left[\frac{1}{x^2y^2}\begin{pmatrix} x^2(2-x-y)\\ y^2(4x-x^2-\sin x)\end{pmatrix}\right]$$

$$= \frac{\partial}{\partial x}\frac{1}{y^2}(2-x-y) + \frac{\partial}{\partial y}\frac{1}{x^2}(4x-x^2-\sin x)$$

$$= -\frac{1}{y^2} < 0$$

Example.

$$\dot{x} = y$$

$$\dot{y} = -x - y + x^2 + y^2$$

Show there is no closed orbits.

Solution: $g(x,y) = e^{-2x}$

$$\nabla \cdot (g(\underline{x}) \cdot \dot{\underline{x}}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \begin{bmatrix} e^{-2x} \cdot y \\ e^{-2x}(-x-y+x^2+y^2) \end{bmatrix}$$

$$= -2ye^{-2x} - e^{-2x} + e^{-2x} y \cdot 2 =$$

$$= -e^{-2x} < 0$$

no closed orbits

Poincare - Bendixson Theorem

$\dot{\underline{x}} = \underline{f}(\underline{x})$

① R is closed and bounded subset of R^2

↓ includes its boundary

↓ does not go to ∞

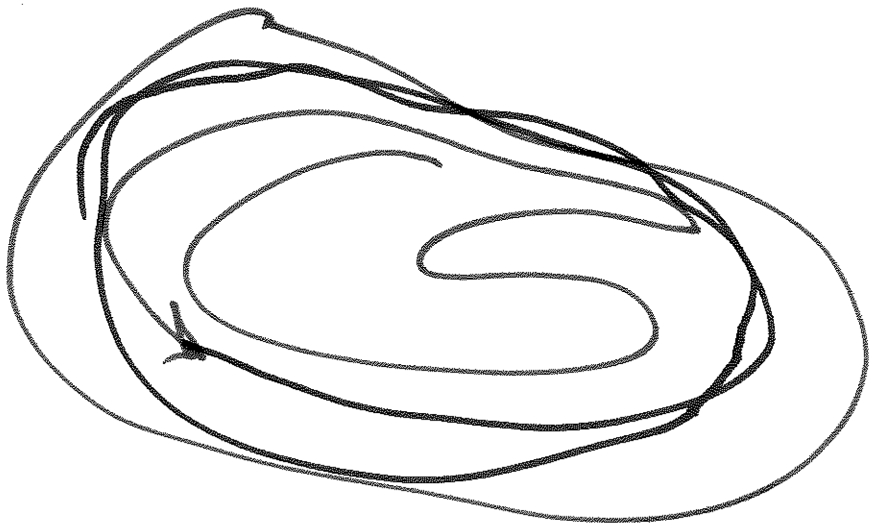
② Let $\underline{f}(\underline{x})$ be continuously differentiable on an open set that includes R

③ R does NOT have any fixed points

④ There exists a trajectory C that is confined in R.

Then C is either a closed orbit or spirals towards a closed orbit.

$$\underline{\dot{x}} = \underline{f}(\underline{x})$$

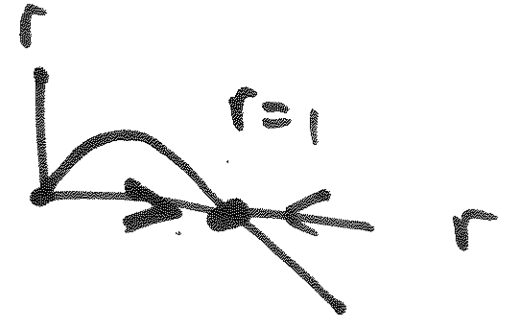


Consider: $\dot{\theta} = 1$

$$\dot{r} = r(1-r^2) + \mu r \cos\theta$$

Show that there is a limit cycle for $\mu > 0$ as long as μ is small enough.

<u>Solution</u> if $\mu = 0$:



stable limit cycle.

Consider $\mu > 0$.

Find R^- such that if $r < R^-$ then $\dot{r} > 0$

<u>Want</u> $r(1-r^2) + \mu r \cos\theta > 0$ for any θ

Worst case: $\cos\theta = -1$

$$r(1-r^2) - \mu r > 0, \quad r > 0,$$

$$1 - r^2 - \mu > 0, \quad 1 - \mu > r^2, \quad r < \sqrt{1-\mu}$$

IF $r < \sqrt{1-\mu}$

Then $\dot{r}(t) > 0$.

Choose $R^- < \sqrt{1-\mu}$

$$R^- = \frac{\sqrt{1-\mu}}{2} \qquad \mu < 1$$

Find R^+ such that

if [redacted] $> R^+$ then $\dot{r}(t) < 0$

$$\dot{r} = r(1-r^2) + \mu r \cos\theta < 0$$

$$r(1-r^2) + \mu r < 0 \qquad R^+ = 2\sqrt{1+\mu}$$

$$1 - r^2 + \mu < 0$$

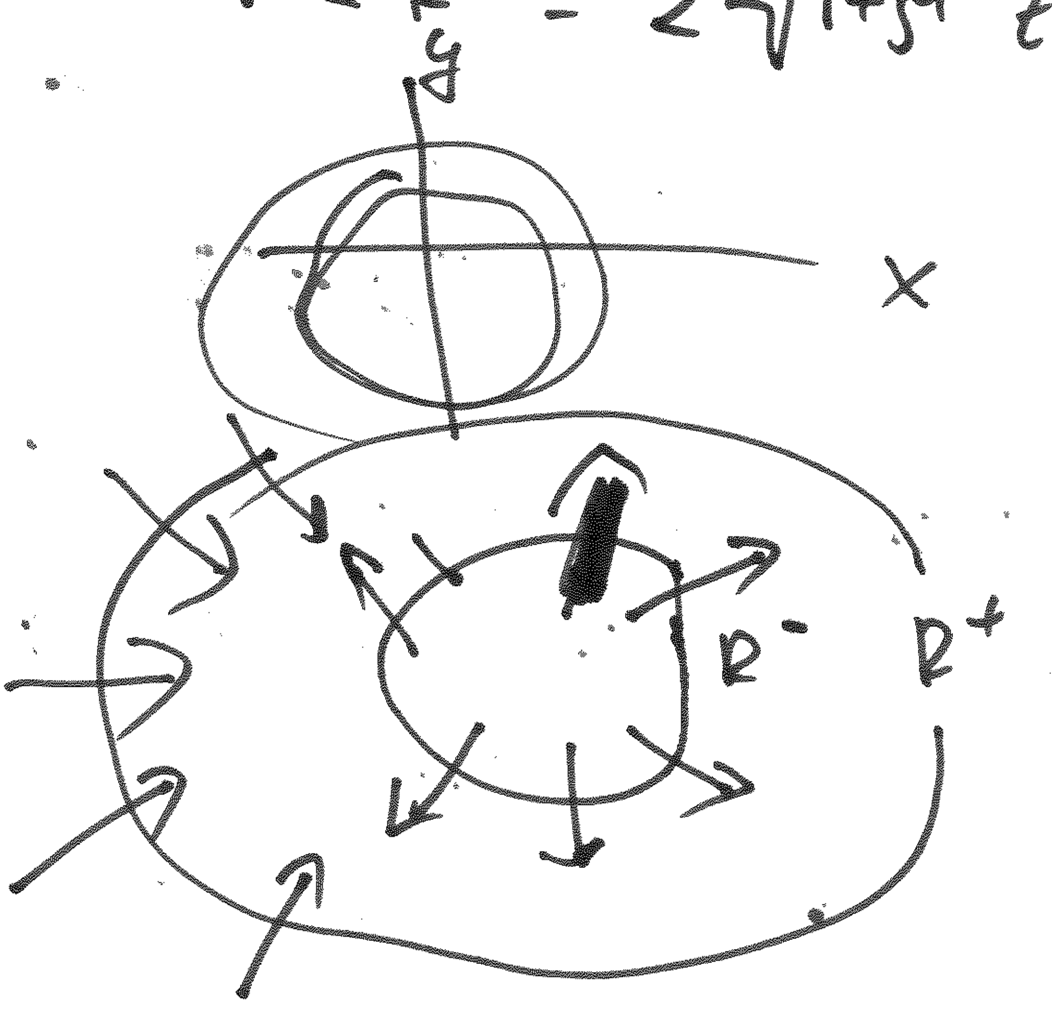
$$1 - r^2 + \mu < 0, \quad r^2 > 1 + \mu, \quad r > \sqrt{1+\mu}$$

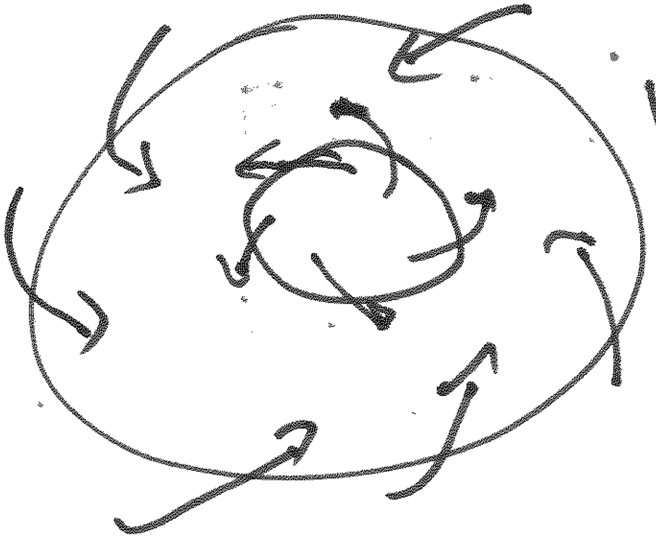
$\dot{\theta} = 1$

$\dot{r} = r(1-r^2) + \mu r \cos\theta$

IF $r > R^- = \frac{\sqrt{1-\mu}}{2}$ then $\dot{r} > 0$

$r < R^+ = 2\sqrt{1+\mu}$ then $\dot{r} < 0$





$R^- < r < R^+$

trapping

region

Example

$$\dot{x} = -x + ay + x^2 y$$

$$\dot{y} = b - ay - x^2 y \qquad \begin{matrix} a > 0 \\ b > 0 \end{matrix}$$

nullcline: $\dot{x} = 0$, vertical: $x = y(a + x^2)$

$$y(x) = \frac{x}{a + x^2}$$

$\dot{y} = 0 \Rightarrow b = y(a + x^2)$

$$y(x) = \frac{b}{a + x^2}$$