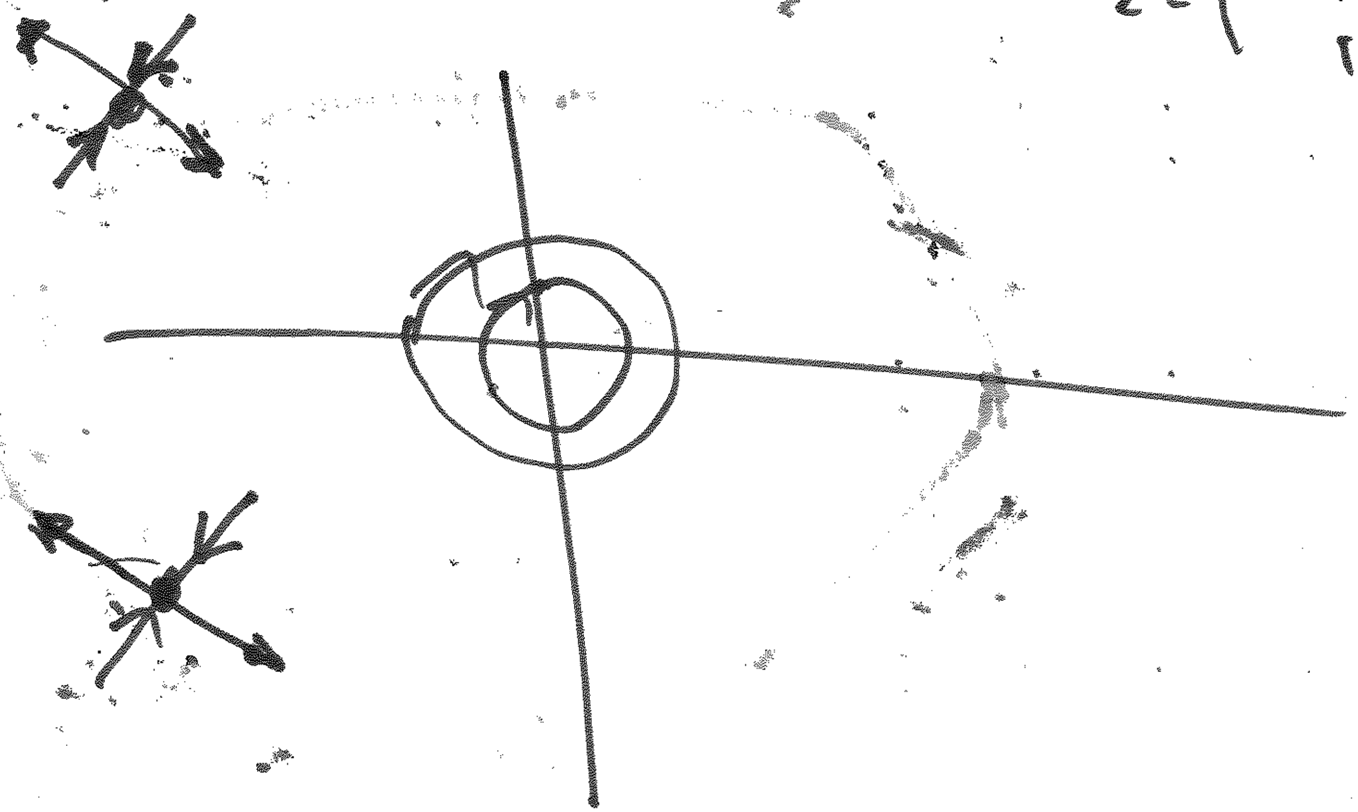
$$\left.\begin{aligned}\dot{x} &= y - y^3 \\ \dot{y} &= -x - y^2\end{aligned}\right\}$$

$x = y = \emptyset$, center

$(-1, 1)$ $\quad \lambda_{1,2} = -1 \pm \sqrt{3}$ $\quad v_1 = \begin{pmatrix} -1 \pm \sqrt{3} \\ 1 \end{pmatrix}$

$(-1, -1)$ $\quad \lambda_{12} = 1 \pm \sqrt{3}$ $\quad v_2 = \begin{pmatrix} 1 \mp \sqrt{3} \\ 1 \end{pmatrix}$

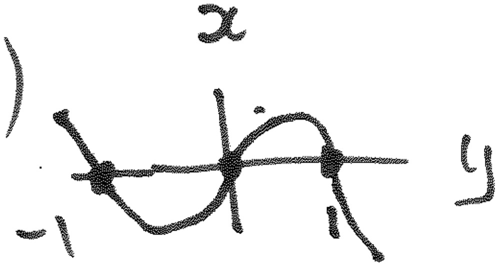
1



$\dot{x} = y - y^3 = y(1-y^2)$

$\dot{y} = -x - y^2$

↳ $x = -y^2$

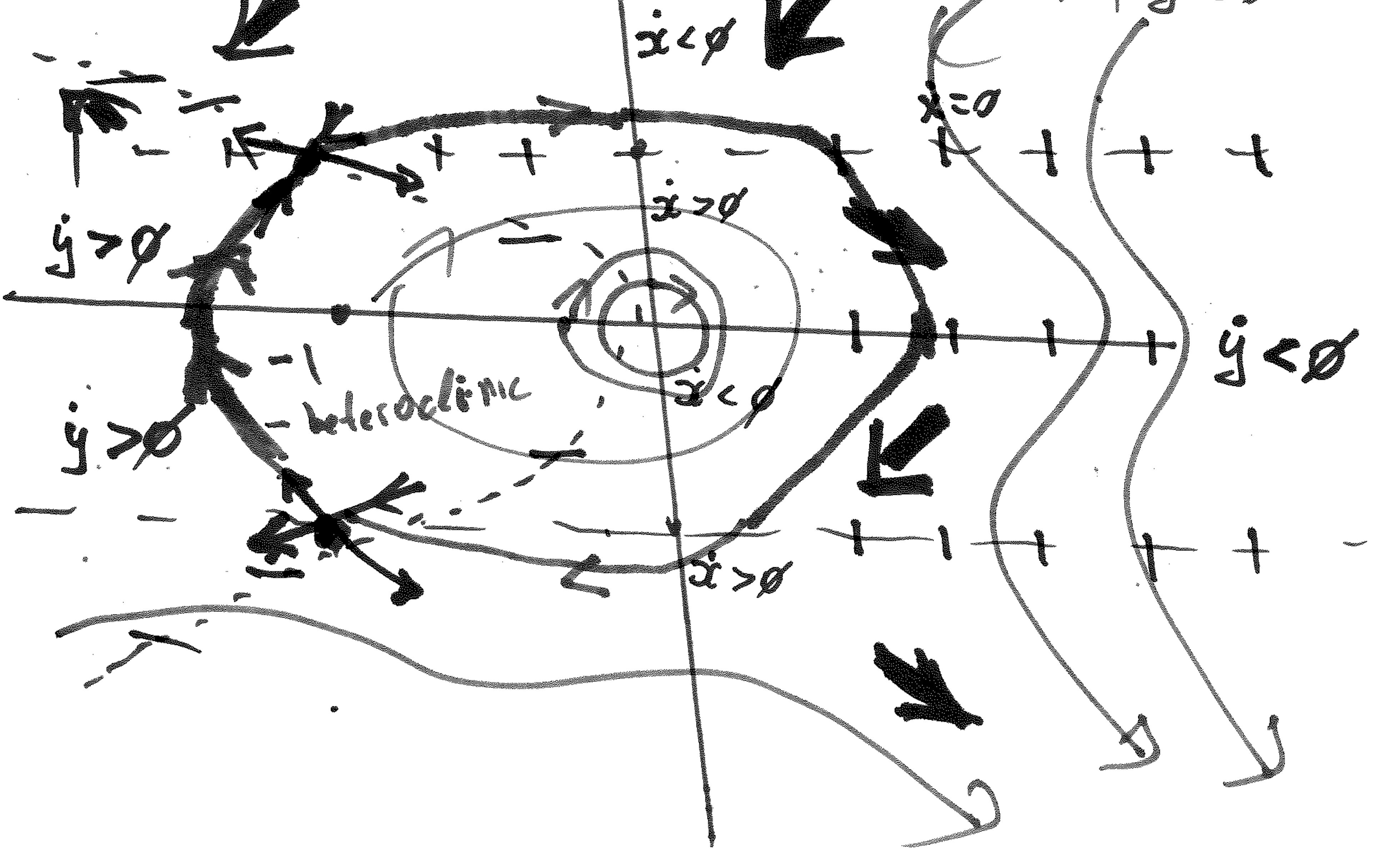


$\dot{x} < 0,\ y > 1$

$\dot{x} > 0$ $\quad 0 \le y \le 1$ or $y < -1$

$\dot{x} < 0,\ -1 < y < 0$

?



Ex

$\dot{x} = y$

$\dot{y} = x - x^2$

Show that there is a homoclinic orbit by using reversibility of a system

Solution

Fixed point $x^* = y^* = \emptyset$; $J(x,y) = \begin{pmatrix} \emptyset & 1 \\ 1-2x & \emptyset \end{pmatrix}$

$J(\emptyset,\emptyset) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\lambda_{1,2} = \dfrac{0 \pm \sqrt{4}}{2} = \pm 1$

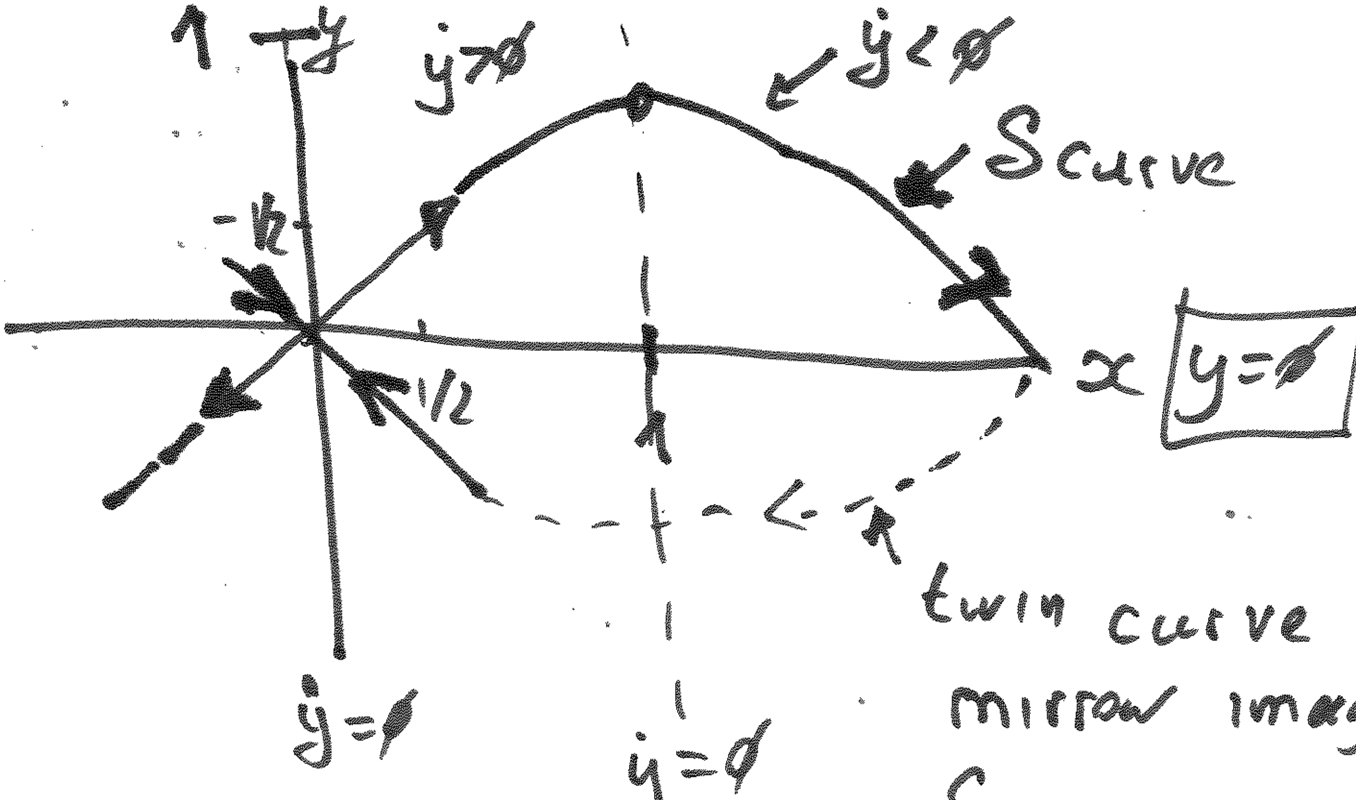
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a = b \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \lambda_1 = 1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -\begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow d = -c \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \lambda_2 = -1$

$\dot{x} = y$ $\quad \lambda_1 = 1;\ v_1 = \begin{pmatrix}1\\1\end{pmatrix};\ \lambda_2 = -1;\ v_2 = \begin{pmatrix}1\\-1\end{pmatrix};$

$\dot{y} = x - x^2 = x(1-x);$

$\dot{x} = f(x,y) = y \rightarrow$ is odd in y
$\dot{y} = g(x,y) = x - x^2$ is even in y
$\Big]$ system is reversible



twin curve is a mirror image of S curve

$$\dot{x} = -2\cos x - \cos y$$
$$\dot{y} = -2\cos y - \cos x = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} \cos x \\ \cos y \end{pmatrix}$$

Show that the system is reversible but not conservative, plot phase portrait.

<u>Solution</u>

$$t = -\tau$$
$$x = -\mu(\tau)$$
$$y = -\nu(\tau)$$

$$\dot{x} = \frac{dx}{dt} = \frac{d\mu}{d\tau}(-1)\cdot(-1) = -2\cos(-\mu) - \cos(-\nu)$$
$$= -2\cos\mu - \cos\nu$$
$$\frac{d\mu}{d\tau} = -2\cos\mu - \cos\nu$$

$$\dot{y} = \frac{dy}{dt} = (-)(-1)\cdot\frac{d\nu}{d\tau} = -2\cos(-\nu) - \cos(-\mu)$$
$$= -2\cos\nu - \cos\mu$$

Find Fixed points

$$2\cos x^* + \cos y^* = 0$$
$$\cos x^* + 2\cos y^* = 0$$

$$\cos x^* = -\frac{1}{2}\cos y^*$$
$$\cos x^* = -2\cos y^*$$

$$\Rightarrow \quad +\frac{1}{2}\cos y^* = +2\cos y^*$$
$$\cos y^* = 4\cos y^*$$
$$\cos y^* = 0 = \cos x^*$$
$$x^* = y^* = \frac{\pi}{2} + \pi \cdot n$$

$n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$$J(x,y) = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}$$

$x = y = \pi/2$: $J\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\lambda_1 = 3$; $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$

$\lambda_2 = 1$; $\begin{pmatrix} -1 \\ 1 \end{pmatrix} = v_2$

$x = \pi/2$; $y = -\pi/2$: $J\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$;

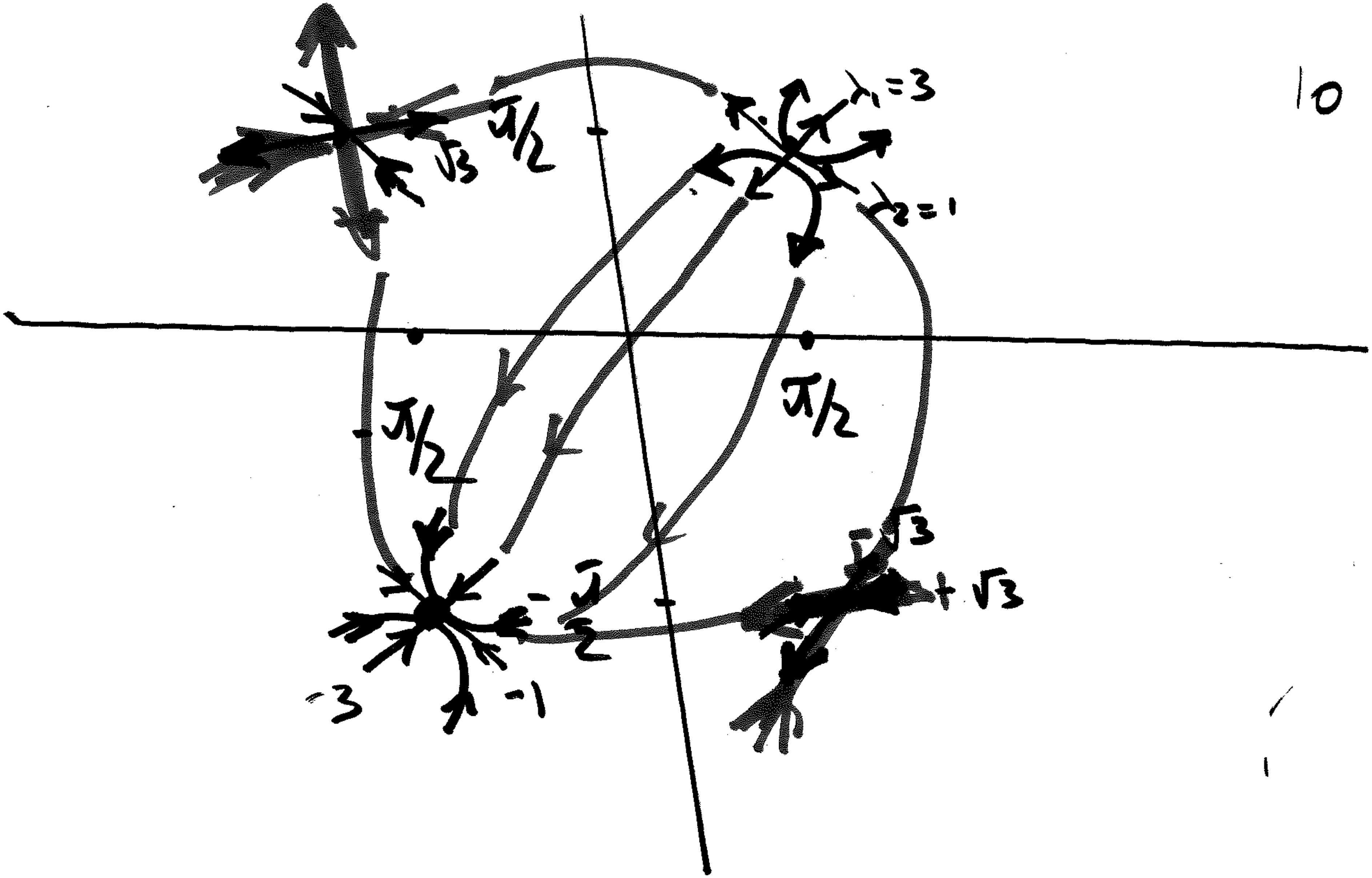
$$\boxed{\begin{matrix} -\sqrt{3} & \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} \\ \sqrt{3} & \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} \end{matrix}}$$

$x = -\frac{\pi}{2}$; $y = -\frac{\pi}{2}$: $J\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

$$\boxed{\begin{matrix} -3 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -1 & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{matrix}}$$

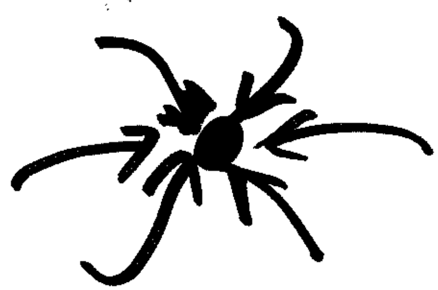
$x = -\frac{\pi}{2}$; $y = \frac{\pi}{2}$: $J\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$, $\lambda_1 = -\sqrt{3}$ $\begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} = v_1$

$\lambda_2 = \sqrt{3}$ $v_2 = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$


λ₁=3
λ₂=1
π/2
√3
π/2
π/2
-π/2
-√3
+√3
-3
-1

System with attractive fixed can not be conservative. If there is a fixed point, then a region around fixed point is a locus of attraction.

Conserved quantity would be constant on that set. Therefore it can not be conserved quantity.



Pendulum $\ddot{\theta} + \frac{g}{L}\sin\theta = \emptyset$

$t = T \cdot \tau$

$\cancel{\frac{d}{dt} = \frac{1}{T}\frac{d}{d\theta}}$ $\frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}$

$\frac{d^2}{d\tau^2}\theta(\tau) + T^2\frac{g}{L}\sin\theta = \emptyset$

$T = \sqrt{L/g}$

$\ddot{\theta} + \sin\theta = \emptyset$

$$\ddot{\theta} + \sin\theta = 0$$

$$\omega = \dot{\theta}$$

$$\dot{\omega} = \ddot{\theta} = -\sin\theta$$

$$\frac{d}{d\tau}\begin{pmatrix}\theta(\tau)\\ \omega(\tau)\end{pmatrix} = \begin{pmatrix}\omega(\tau)\\ -\sin\theta\end{pmatrix} \Rightarrow$$

$(0, 0)$

$(\pi, 0)$

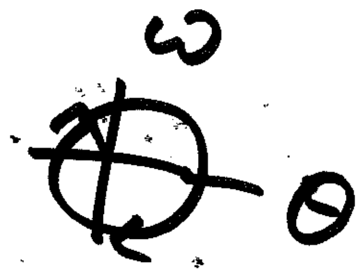
$(-\pi, 0)$

$$(n\pi, 0)$$

$$n = 0, \pm 1, \pm 2 \ldots$$

$$J(\theta,\omega)=\begin{pmatrix}0 & 1\\ -\cos\theta & 0\end{pmatrix}$$

$$J(0,0)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$ – center



$$J(\pi,0)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$\lambda_1 = 1$, $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$

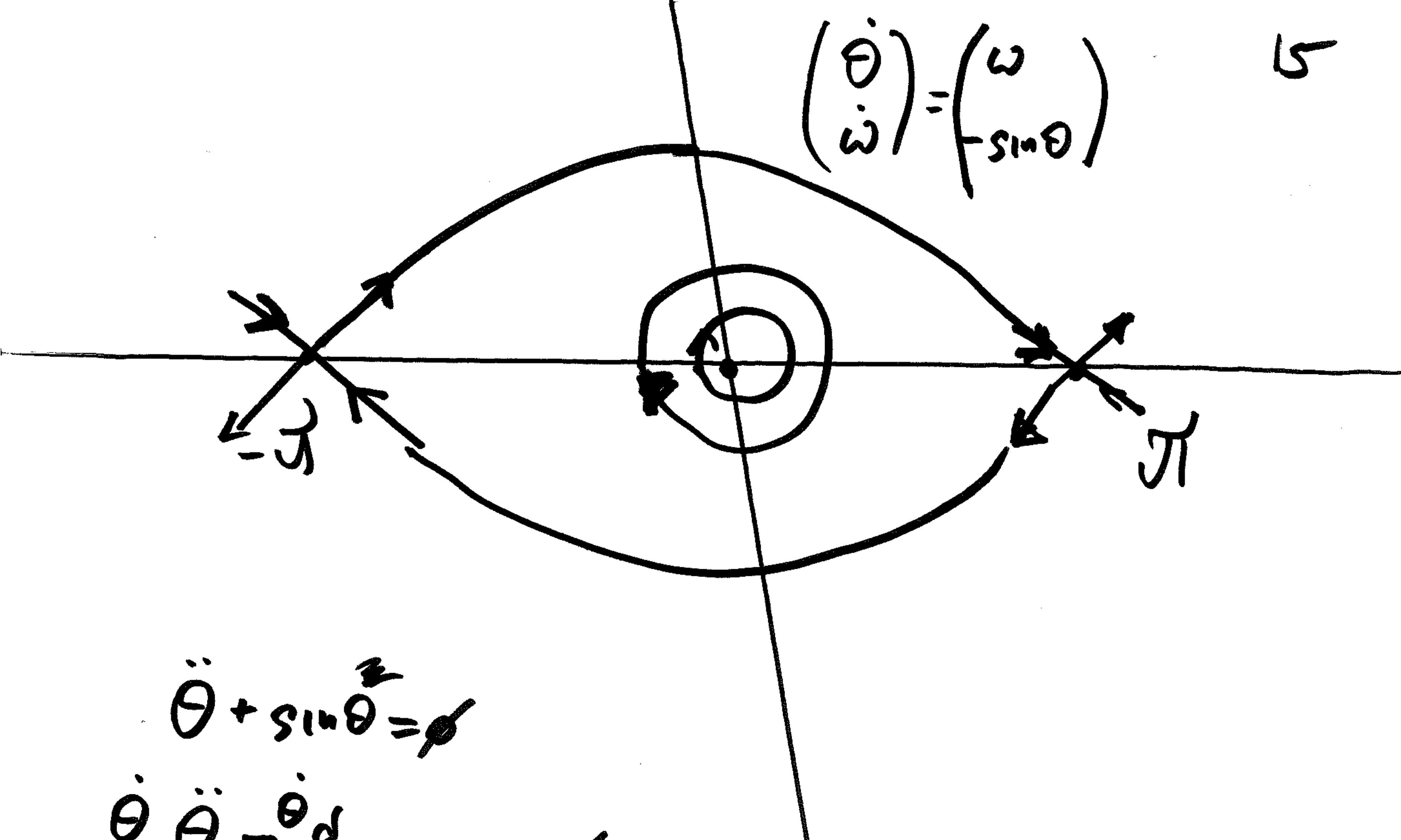
$\lambda_1 = -1$ $v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$J(-\pi,0) = J(\pi,0)$

$$J(\pi,0) = J(-\pi,0) = J(3\pi,0) = \dots$$

$$J(0,0) = J(2\pi,0) = J(4\pi,0) = \dots$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ -\sin\theta \end{pmatrix}$$



$$\ddot{\theta} + \sin\theta = 0$$

$$\dot{\theta}\,\ddot{\theta} - \dot{\theta}\frac{d}{dx}\cos x = 0$$