A

(*) $\frac{d}{dt} x(t) = f(x(t))$ - autonomous first order ordinary differential equation.

Fixed point x^* such that $f(x^*) = \emptyset$;

Change of variables

(**) $x(t) = x^* + \eta(t)$; $|\eta(t)| << 1$;

(**) => (*)

$$\frac{d}{dt} \eta(t) = f(x^* + \eta(t)) \approx$$

$$\approx f(x^*) + f'(x^*)\eta(t) + f''(t)\,\eta^2/2 + f'''($$

$$\frac{d}{dt}\eta(t) = f(x^*) + f'(x^*)\,\eta(t) + f''(x^*)\,\eta^2(t)/2 + f'''(x^*)\,\eta^3/6 + \ldots$$

$f(x^*) = \emptyset$, because x^* is a fixed point

$$\frac{d}{dt}\eta(t) = f'(x^*)\,\eta(t);$$

$$\eta(t) = \eta(t=\emptyset)\,e^{f'(x^*)t};$$

- if $f'(x^*) < \emptyset$, then $\lim\limits_{t\to\infty} \eta(t) = \emptyset$; stable fixed point

x^* is a stable fixed point iff $\forall \varepsilon, |\varepsilon| \ll 1$, $x(t=\emptyset) = x^* + \epsilon \Rightarrow \lim\limits_{t\to\infty} x(t) = x^*$

C

- $f'(x^*) > \emptyset$,

$$\lim_{t \to \infty} \eta(t) = \infty$$

unstable fixed point.

IF. $x(t = \emptyset) = x^*$ then

$$\lim_{t \to \infty} x(t) \neq x^*$$

$$\lim_{t \to -\infty} x(t) = x^*$$

D

- $f'(x^*) = \emptyset;$

$$\frac{d}{dt}\eta(t) = f'(x^*)\,\eta(t) = \cancel{\emptyset}$$

$$\eta(t) = \eta(t = \emptyset)$$

need to write more terms in T

$$\frac{d\eta(t)}{dt} = f''(x^*)\,\eta^2(t)/2$$

$$\int \frac{d\eta(t)}{\eta^2(t)} = \int dt\, f''(x^*)/2 = \frac{f''(x^*)}{2}\,t + C$$

$$-\frac{1}{\eta(t)} = f''(x^*)\,t/2 + C \Rightarrow \cancel{\eta(t) = \frac{1}{C + f''(x^*)}}$$

$$z(t) = \frac{1}{c + f''(x^*)\,t/2} \Rightarrow$$ analyze in more details.

F

Consider $\dot{x}(t) = \sin(x(t))$;

Find all fixed points and classify their stability.

$x = n\pi$; $n = \emptyset, \pm 1, \pm 2, \pm 3 \ldots$

$x(t) = f(x(t)) = \sin(x(t))$;

$f'(x) = \cos x$;

$\cos(\pi n) = (-1)^n$

$e^{ix} = \cos x + i \sin x$

$\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2}$

n is even, $n = 2k$, $k = \emptyset, \pm 1, \pm 2 \ldots$

$\cos(\pi n) = (-1)^{2k} = 1 > \emptyset$ unstable.

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n is odd

$n = 2k+1$, $\cos(\pi n) = (-1)^{2k+1} = (-1) < 0$

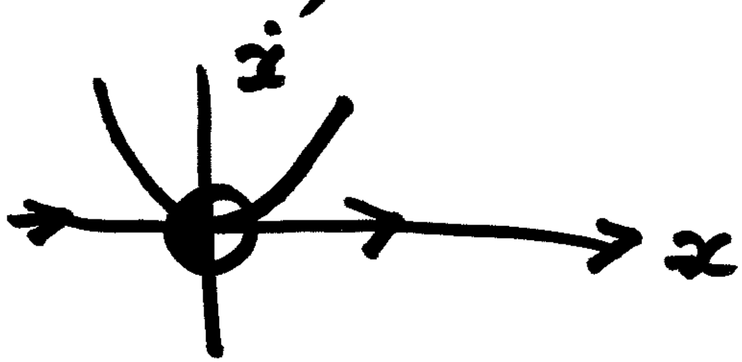
Stable

$x^* = n\pi$, n is odd, stable

n is even unstable

$\dot{x}(t) = x^2$

$x^* = \emptyset;$



mixed stability:

$\dot{x} = f(x) = x^2;$

$f'(x) = 2x,$

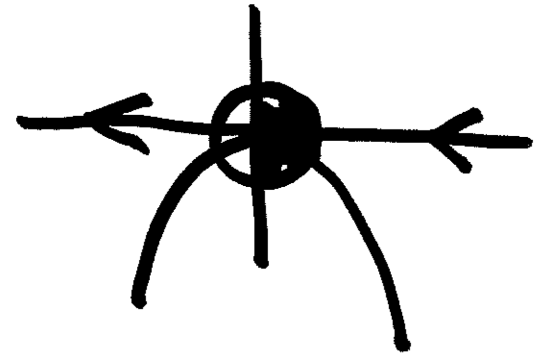
$f'(x^*) = \emptyset;$

stable from the left

unstable from the right

;

$\dot{x}(t) = -x^2$ H



unstable on the left

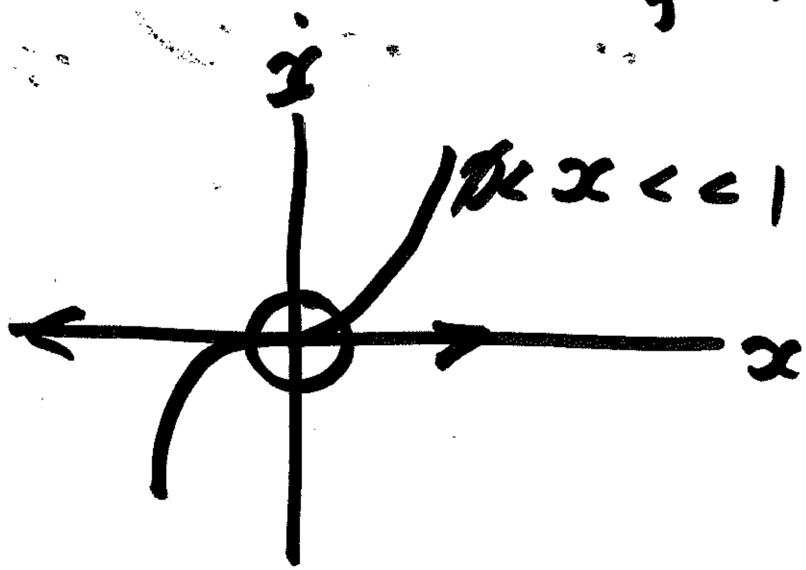
stable on the right.

I

$\dot{x}(t) = x^3(t)$ $\quad\quad$ $\dot{x}(t) = -x^3(t)$

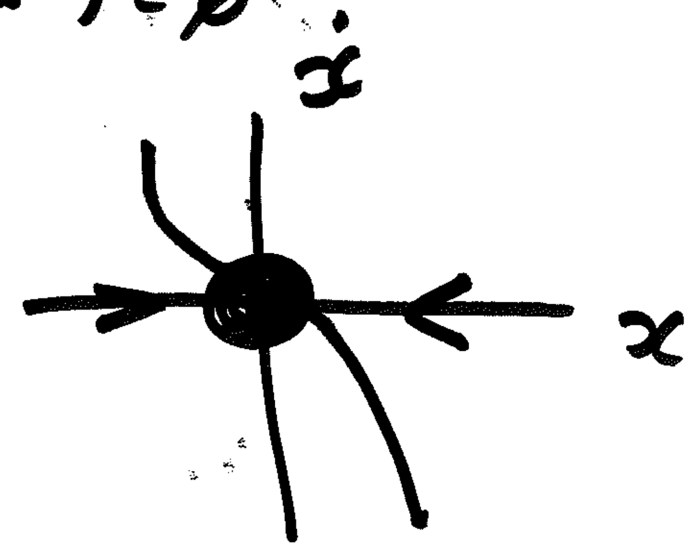
$x^* = \emptyset$ is a fixed point

$f'(x^*) = f''(x^*) = \emptyset$



$x^* = \emptyset$ is unstable fixed point

$$\int \frac{dx}{x^3} = \frac{x^{-2}}{-2}$$



$x^* = \emptyset$ is stable fixed point

J

$\dot{x} = x^n$,

$n = 1, 2, 3, 4, \ldots$

n is odd - x^* is either

[redacted]

unstable

n is even

unstable on the right

stable on the left

• $\dot{x} = -x^n \rightarrow$ reverse

Population growth $k \to k\left(1 - \frac{N(t)}{K}\right)$ K

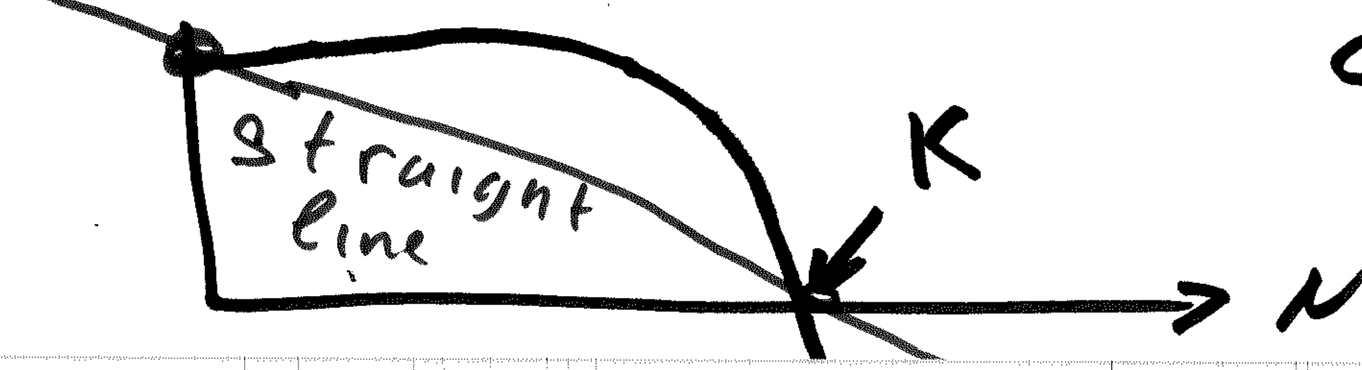
$N(t)$ – population of Susanville

$$\frac{d}{dt} N(t) = k\, N(t), \quad k \text{ is a constant}$$

$$N(t) = N(t=0)\, e^{kt}$$

$k(N)$

K - carrying capacity



L

$$\frac{d}{dt} N(t) = k\, N(t)\left(1 - \frac{N(t)}{K}\right);$$

$$\int \frac{dN}{N\left(1 - \frac{N}{K}\right)} = k \int dt = kt + C$$

$$\frac{1}{N\left(1 - \frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1 - N/K} =$$

$$= \frac{A - AN/K + BN}{N(1 - N/K)}$$

$$A = 1, \quad B = \frac{A}{K} = \frac{1}{K}$$

$$\frac{1}{N(}$$

M

$$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{1}{N} + \frac{1}{K-N}$$

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \ln N - \ln(K-N) =$$

$$= \ln \frac{N}{K-N}$$

$$\cancel{0} < N < K$$

$$\frac{N}{K-N} = e^{kt+C}$$

$$\cancel{N = e^{kt} e^{c} = Ae^{kt}}$$

2

$$N(t) = eK - Ne$$

$$N(t)\left(1+\exp(kt+c)\right) = \exp(kt+c)K$$

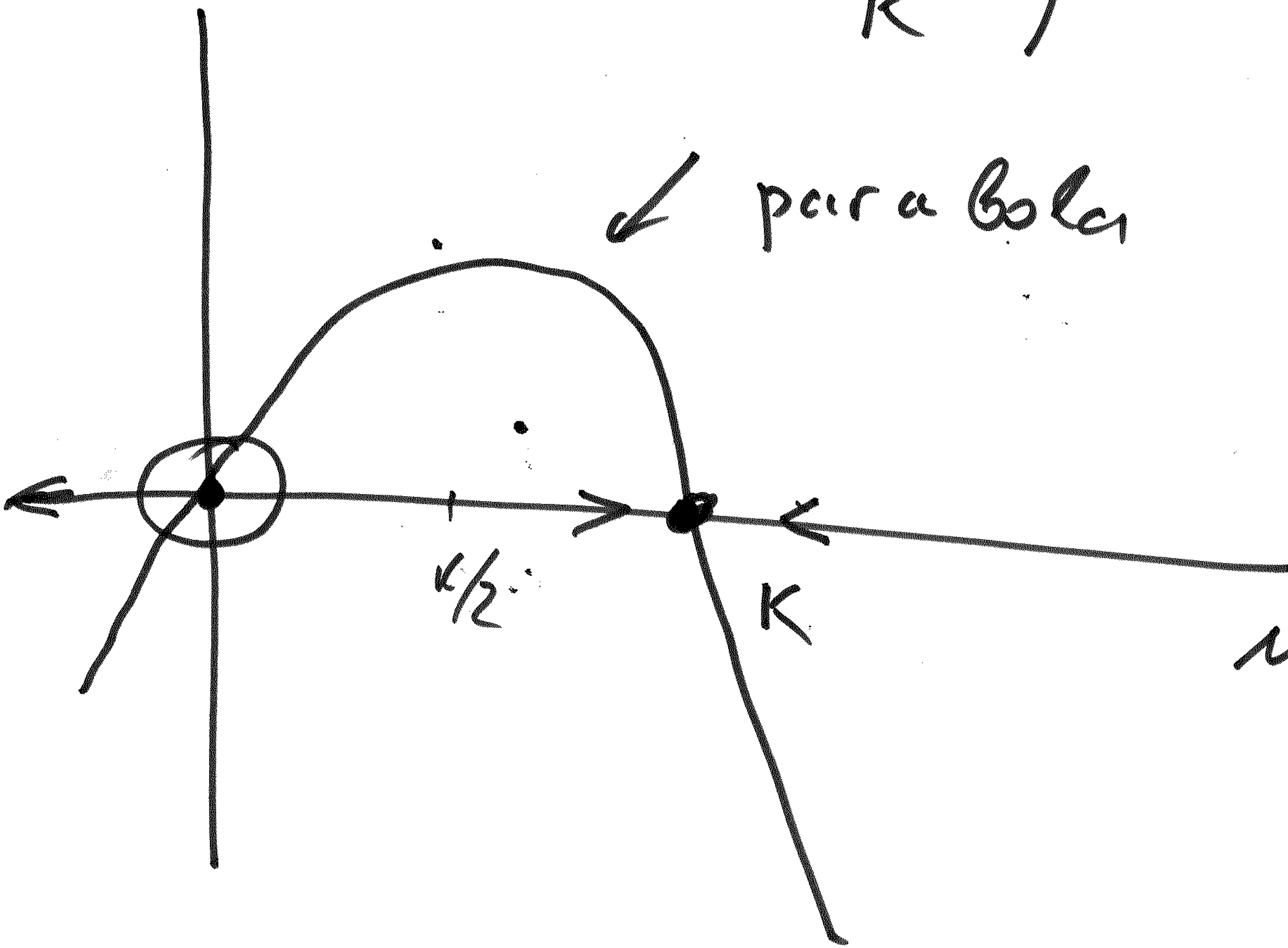
$$N(t) = \frac{K\exp(kt+c)}{1+\exp(kt+c)} = \frac{K}{1+Ce^{-kt}}$$

$$N(t) = \frac{K}{1+e^{-kt}\left(\frac{K}{N_0}-1\right)}$$

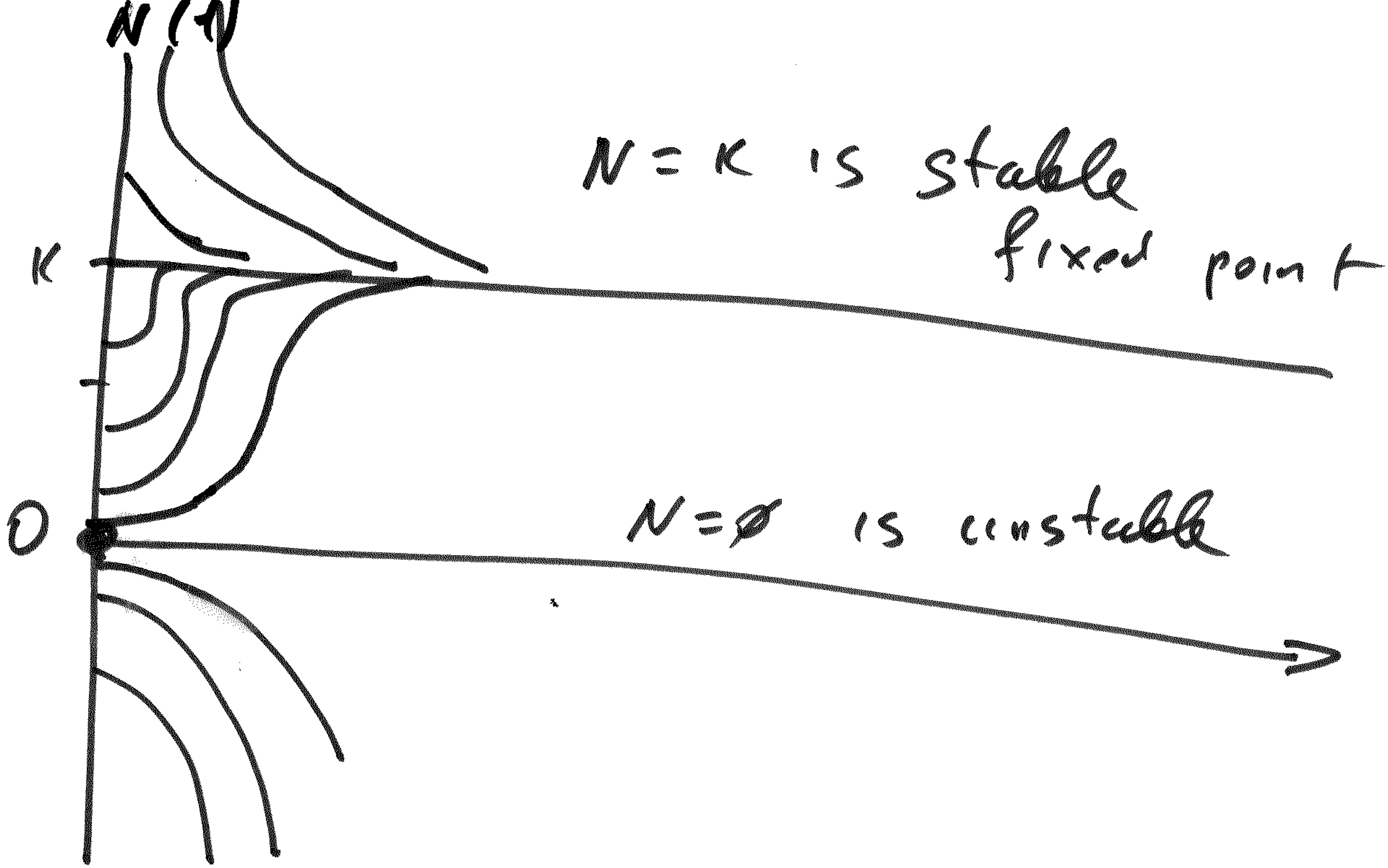
$$N(t=0) = N_0$$

$$\frac{dN(t)}{dt} = kN(t)\left(1 - \frac{N(t)}{K}\right)$$

O



P



Q

$$\frac{dN}{dt} = kN\left(1-\frac{N}{K}\right)$$

$$f(N) = kN(1-N/K)$$

$$f'(N) = \frac{d}{dN}\left(kN - \frac{kN^2}{K}\right) =$$

$$= k - \frac{2kN}{K}$$

$N=\emptyset$: $f'(N=\emptyset) = k > \emptyset$ unstable FP

$N=K$: $f'(N=K) = k - \frac{2kK}{K} = -k < \emptyset$ stable

R

$$\frac{d}{dt} x(t) = f(x(t))$$

$$x(t = t_0) = x_0$$

Existence and uniqueness Theorem

Let $f(x)$ and $f'(x)$ be continiously differentiable functions in a vicinity of x_0.

Then solution of Initial Value problem $\dot{x} = f(x)$, $x(t=0) = x_0$ exists and is unique in a vicinity of x_0.

$x_0 \in R$

1987

5

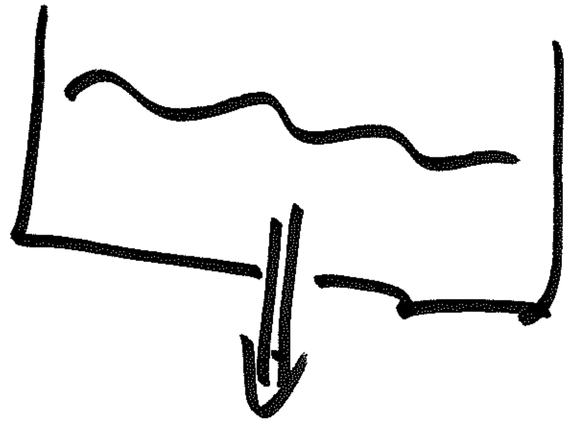
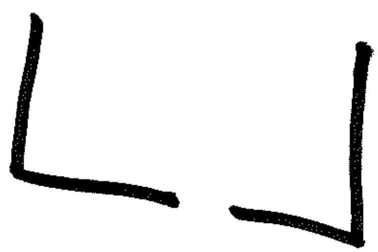
$\leftarrow \ell(t)$ - level of water.

$\dot{\ell}(t) = -k$, k is a constant $k > \emptyset$;

$\emptyset \le \ell(t)$

$$\dot{\ell}(t) = \begin{cases} -k, & \ell(t) > \emptyset \\ 0, & \ell(t) = \emptyset \end{cases}$$

τ

-  $\ell(t)$
-  - no water

Example

$$\frac{d}{dt}\, x(t) = \sqrt{|x(t)|}$$

$$\frac{d}{dt}\, x(t) = f(x(t))$$

$$f(x) = \sqrt{|x|} = \begin{cases} \sqrt{x}, & x > 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

✓

$x > 0,$

$\dot{x} = \sqrt{x}\,;$ $\quad \int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\frac{dx}{dt} = \sqrt{x}\,;\ \int \frac{dx}{\sqrt{x}} = \int dt$

$\int \frac{dx}{\sqrt{x}} = \quad 2\sqrt{x} = t + C$

$(x^n)' = n x^{n-1}$

$x^n = n \int x^{n-1}\,dx$

W

$$2\sqrt{x} = t + c$$

$$\sqrt{x} = \frac{t+c}{2} \qquad \underline{x > \emptyset}$$

$$x(t) = \frac{t^2}{4} + \tilde{c}$$

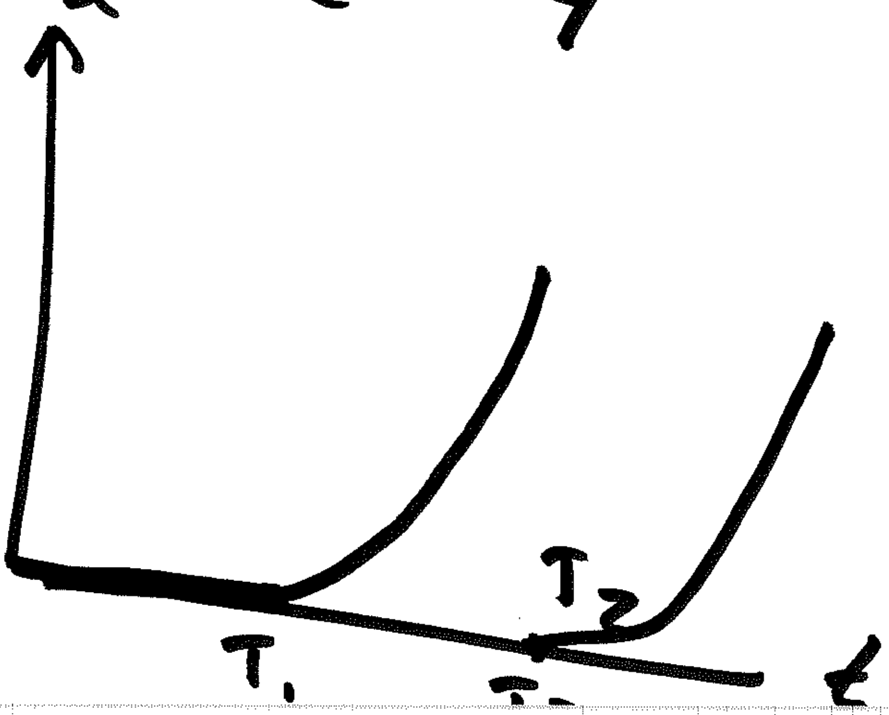
$$x(t=\emptyset) = x_0 > \emptyset$$

$$x(t) = \left[\begin{array}{l} \frac{t^2}{4} + x_0 \\ \\ \emptyset \end{array}\right.$$

$$\dot{x}(t) = \sqrt{|x(t)|}$$

X

$$x(t) = \begin{cases} 0, & 0 < t < T \\ \dfrac{(x-T)^2}{4} & \end{cases}$$



$$\dot{x} = f(x), \quad f(x) = \sqrt{|x|}$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{|x|}}$$

$x = 0$ $\qquad f'(x) \to \infty$

Y

2

$\dot{x}(t) = f(x(t));$

$f(x) > 0$ Right

$f(x) < 0$ Left

$f(x) = 0$ fixed point

oscillations are not possible?