Conservative quantity.

NLS $\quad i\dot{x} + \Delta x + |x|^2 x = \emptyset$

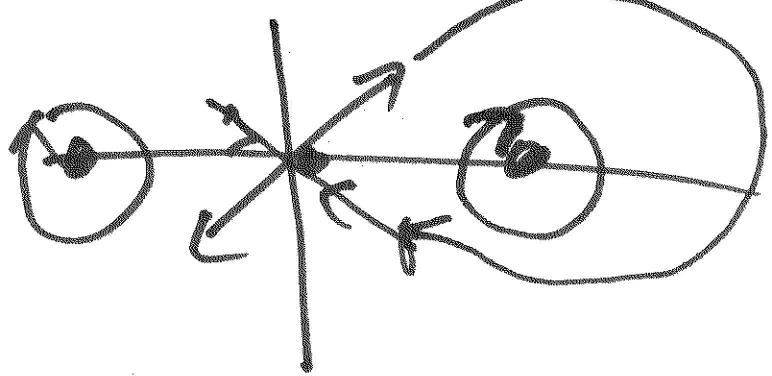
$$x = x(\bar{r})$$

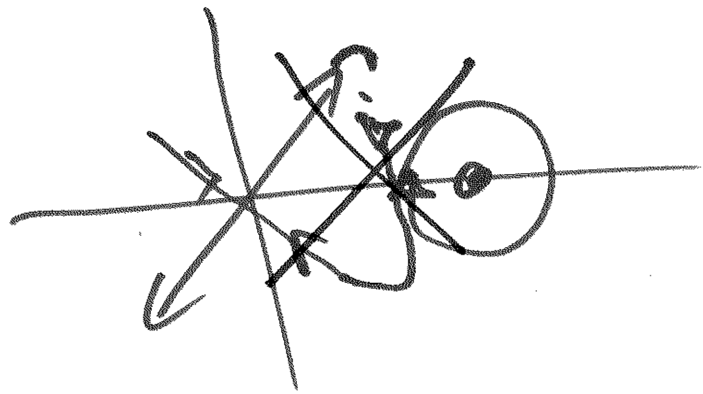
$V(x) = -\frac{1}{2}x^2 + \frac{x^4}{4}$ $\quad \underline{m=1}$

$m\ddot{x} = -\frac{dV}{dx} = x - x^3$

$\ddot{x} = x - \ddot{x}$ $\quad x = \emptyset \leftarrow$ saddle

$x = \pm 1 \leftarrow$ center





$m = 1$

Energy: $E = \frac{1}{2} m \dot{x}^2 + V(x)$

$= \frac{1}{2} \dot{x}^2 + -\frac{1}{2} x^2 + x^4$

Nonlinear center for conservative systems

Let $\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t))$ with $\underline{x} \in R^2$ and $\underline{f}(\underline{x})$ is continuously differentiable.
Let $E(\underline{x})$ be conserved quantity and
Let x^* be a minimum of $E(\underline{x})$.
Suppose that $\underline{x}^*$ is an isolated fixed point of $\underline{\dot{x}} = f(\underline{x})$.
Then all trajectories sufficiently close to x^* are closed.

Example

$$C = E(x, \dot{x}) = -x^2 + \dot{x}^2 + x^4 + x^4 \cdot \dot{x}^4 + \sin(x \cdot \dot{x}^2)$$

$$\varnothing = \frac{d}{dt} C = -2x \cdot \dot{x} + 2x\ddot{x} + 4x^3\dot{x} + 4x^3 \cdot \dot{x}^5 + 4x^4 \dot{x}^3 \cdot \ddot{x} + \cos(x \cdot \dot{x}^2)$$

$$+ \cos(x \cdot \dot{x}^2) \cdot (\dot{x}^3 + 2x\dot{x}\ddot{x}) = \varnothing$$

Time reversible systems

Newton: ~~dx~~

$$m \frac{d^2}{dt^2} x(t) + \frac{d}{dx} V(x) = \emptyset$$

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= v(t) \\ m \frac{d}{dt} v(t) &= -\frac{d}{dx} V(x) \end{aligned}\right\}$$

$$t = -\tau \qquad \frac{d}{dt} = -\frac{d}{d\tau}$$

$$v(t) = -\omega(\tau)$$

$$x(t) = y(\tau)$$

$$\left.\begin{aligned} -\frac{d}{d\tau} y(\tau) &= -\omega(\tau) \\ (-1)(-1) m \frac{d}{d\tau} \omega(\tau) &= -\frac{d}{dx} V(y) \end{aligned}\right\} \quad \left\{\begin{aligned} \frac{d}{d\tau} y(\tau) &= \omega(\tau) \\ m \frac{d}{d\tau} \omega(\tau) &= -\frac{d}{dy} V(y) \end{aligned}\right.$$

$$\left.\begin{aligned} \frac{d}{dt}x(t) &= f(x(t), y(t)) \\ \frac{d}{dt}y(t) &= g(x(t), y(t)) \end{aligned}\right\} (*)$$

Let f be odd in y
g be even in y

Then $(*)$
is time reversible.

f is y-odd: $f(x,-y) = -f(x,y)$

g is y even $g(x,y) = g(x,-y)$

$$t = -\tau, \quad \frac{d}{dt} = -\frac{d}{d\tau}$$

$$y(t) = -Y(\tau)$$

$$x(t) = X(\tau)$$

$$-\frac{d}{d\tau} X(\tau) = f(X(\tau), -Y(\tau)) =$$

$$\frac{d}{d\tau} X(\tau) = f(X(\tau), Y(\tau)) = -f(X(\tau), Y(\tau))$$

$$\cdot \quad \frac{d}{d\tau} Y(\tau)(-1)(-1) = g(X(\tau), -Y(\tau)) = g(X(\tau), Y(\tau))$$

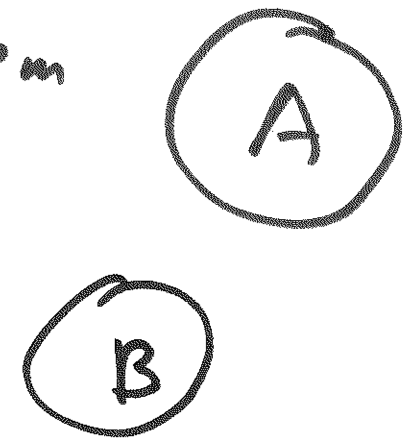
If $f(x,y)$ is y-odd, and $g(x,y)$ is y-even, the system

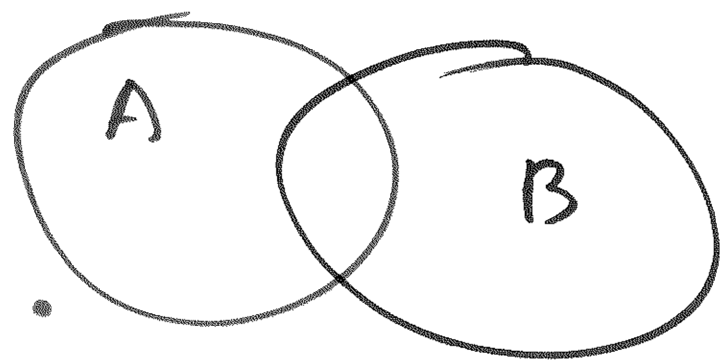
$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

is time reversible.

System is time reversible, if change of variables $t = -\tau$ and some other changes does not change the form of equation conservative system

Time reversible





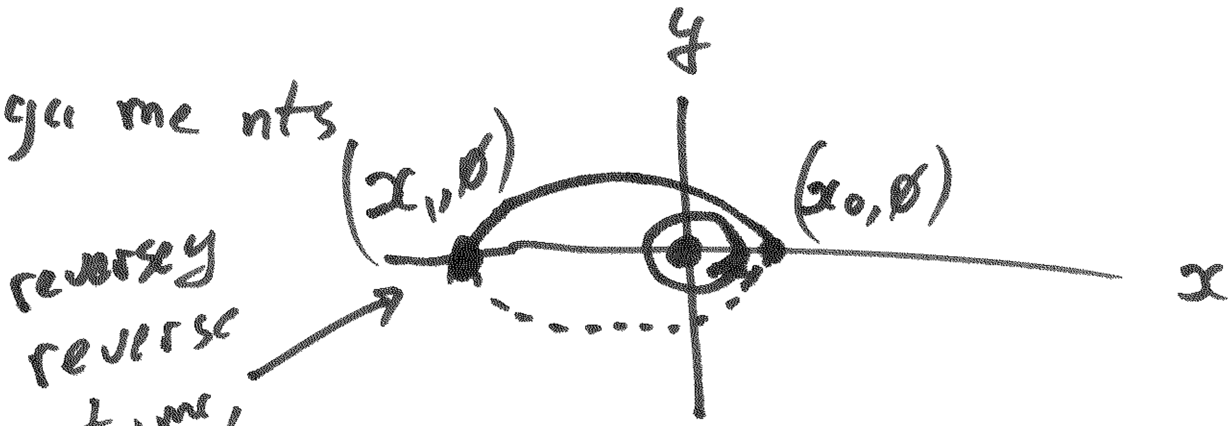
Non linear center

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

And let x^*, y^* be a linear center. Then, if the system is reversible, then all trajectories sufficiently close to x^* y^* are closed.

$x^* = y^* = \emptyset$.

Arguments



Consider

$$\dot{x} = y - y^3 = f(x,y) = y(1-y^2)$$

$$\dot{y} = -x - y^2 = g(x,y)$$

Show that origin is nonlinear center, plot phase portrait

SOLUTION

Reversible if f is y odd, g is y-even

$$f(x,-y) = -y + y^3 = -f(x,y)$$

$$g(x,-y) = -x - y^2 = g(x,y)$$

$$J = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$$

Fixed points: $y = x = \emptyset$;

$y = \pm 1$

$x = -y^2 = -1$

$(\emptyset, \emptyset)$

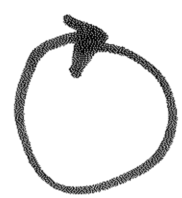
$(-1, -1)$

$(-1, 1)$

$J(\emptyset,\emptyset) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad T=\emptyset$, $D=1$ center

Satisfy the theorem.

$(\emptyset,\emptyset)$ is a nonlinear center



$(-1,-1)$

$$J(-1,-1) = \begin{pmatrix} 0 & 1-3y^2\big|_{y=-1} \\ -1 & -2y\big|_{y=-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix} \quad \begin{array}{l} Tr = 2 \\ Det = -2 \end{array} \qquad \lambda_{12} = \frac{Tr \pm \sqrt{Tr^2 - 4Det}}{2}$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+8}}{2} =$$

$$= 1 \pm \sqrt{3} \lessgtr 0$$

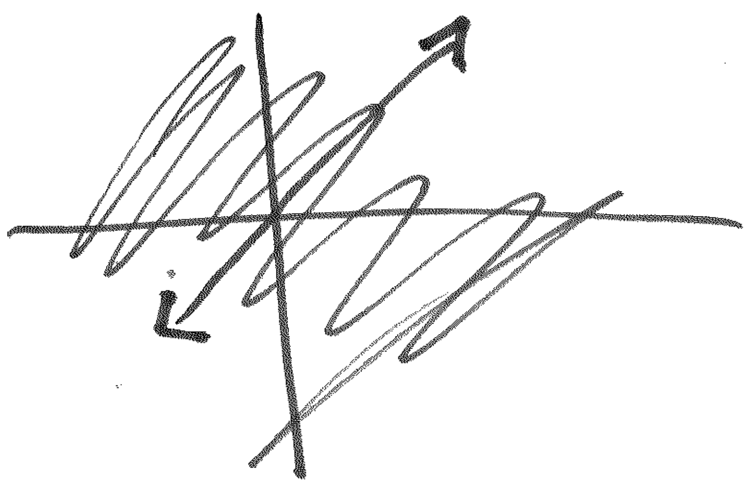
$$\begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1+\sqrt{3}\begin{pmatrix} a \\ b \end{pmatrix}; \quad -2b = (1+\sqrt{3})a$$

$$v_1 = \begin{pmatrix} -2 \\ 1+\sqrt{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = (1-\sqrt{3}) \begin{pmatrix} c \\ d \end{pmatrix}$$

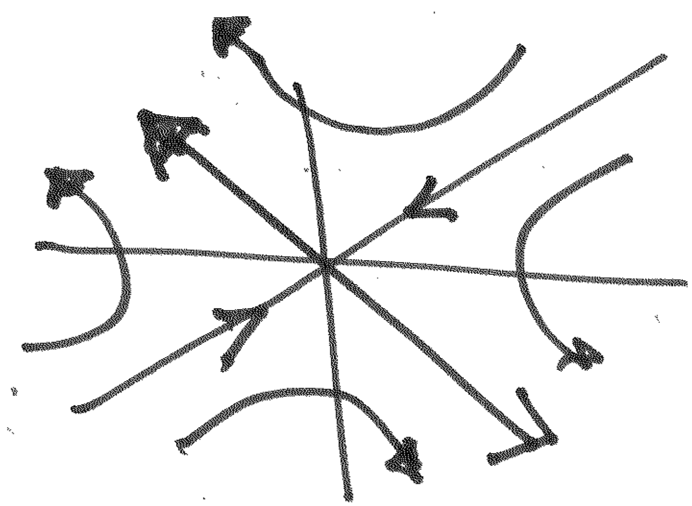
$$-2d = (1-\sqrt{3})c$$

$$v_2 = \begin{pmatrix} -2 \\ 1-\sqrt{3} \end{pmatrix}$$



$\lambda_1 = 1+\sqrt{3}$; $v_1 = \begin{pmatrix} -2 \\ 1+\sqrt{3} \end{pmatrix}$ $(-1, -1)$

$\lambda_2 = 1-\sqrt{3}$; $v_2 = \begin{pmatrix} 2 \\ \sqrt{3}-1 \end{pmatrix}$



$(-1,1)$

$$J(-1,1) = \begin{pmatrix} 0 & 1-3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$$

$Tr = -2, \ \Delta = -2;$

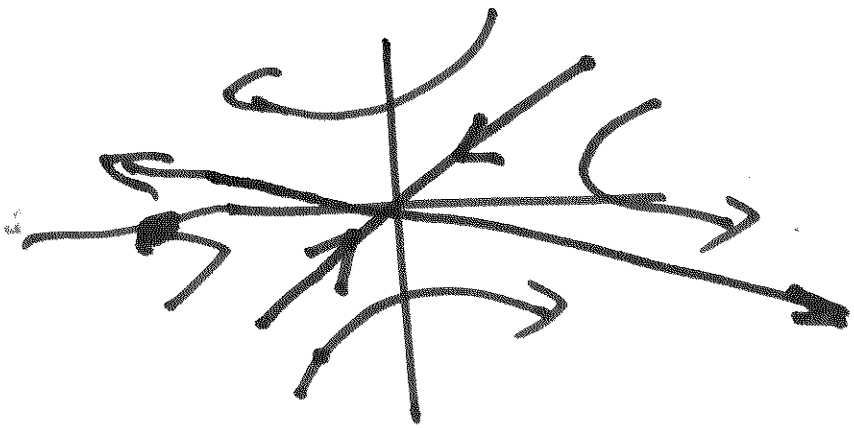
$$\lambda_{12} = \frac{Tr \pm \sqrt{Tr^2 - 4Det}}{2} =$$

$$= -1 \pm \frac{1}{2}\sqrt{4+8} = -1 \pm \sqrt{3}$$

$\lambda_1 = -1-\sqrt{3}; \quad -2b = (-1-\sqrt{3})a; \quad v_1 = \begin{pmatrix} +2 \\ +1+\sqrt{3} \end{pmatrix}$

$\lambda_2 = -1+\sqrt{3} \quad -2b = (-1+\sqrt{3})a$

$v_2 = \begin{pmatrix} -2 \\ -1+\sqrt{3} \end{pmatrix}$



$\dot{x} = y - y^3$

$\dot{y} = -x - y^2;$