$\dot{x} = x(3 - x - 2y)$; Simple growth

$\dot{y} = y(2 - x - y)$; Competition

Interactions

$x = 0 = y$

$x = 0, \; y = 2$

$y = 0, \; x = 3$

$()$

$() \Rightarrow 3 - x - 2y = 0$

$2 - x - y = 0$

$x = 3 - 2y$

$2 - 3 + 2y - y = -1 + y = 0, \; y = 1$

$x = 1$

$(0, 2)$

$(3, 0)$

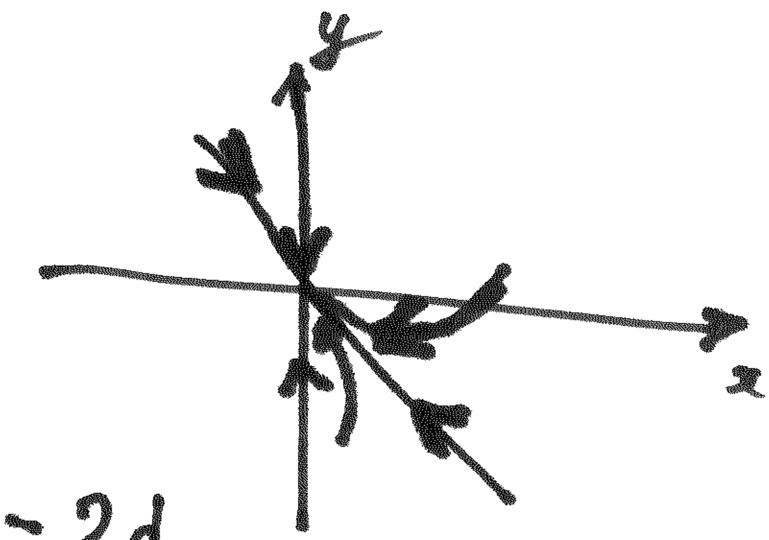
$(1, 1)$

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial g \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

- $(\emptyset, 2)$

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

$\lambda_1 = -1, \qquad \lambda_2 = -2$



$-a = -a$
$-2a - 2b = -b$
$b = -2a$
$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$-c = -2c$
$-2c - 2d = -2d$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Rabbits
win

4

- $(3, \emptyset)$

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

$\lambda_1 = -3$ $\qquad\qquad$ $\lambda_2 = -1$

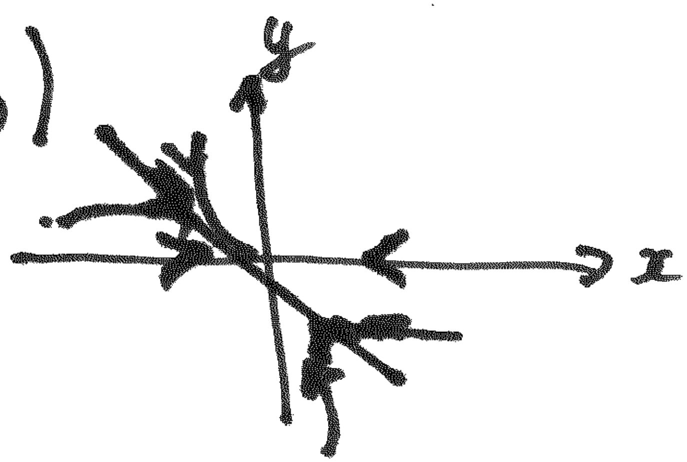
$-3a - 6b = -3a, \ b = \emptyset$ $\qquad$ $-3c - 6d = -c,$

$-b = 3b$ $\qquad$ $-d = -d,$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $2c = -6d$

$c = -3d$

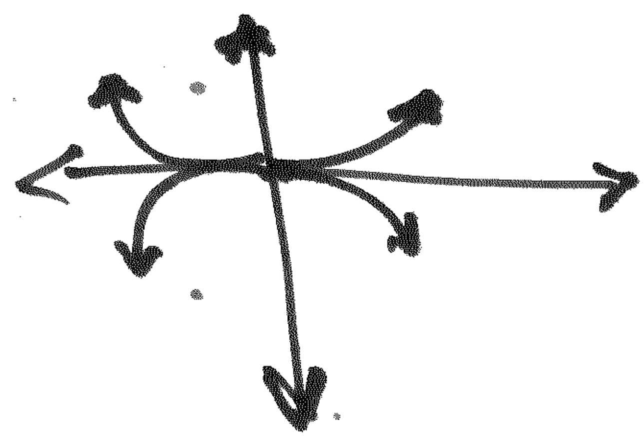
$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$



$(0,0)$; $\quad J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

$\lambda_1 = 2, \qquad \lambda_2 = 3$

$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$(1,1)$

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$(-1-\lambda)^2 - 2 = 0$$

$$(\lambda+1)^2 = 2$$

$$\lambda + 1 = \pm\sqrt{2}$$

$$\lambda = -1 \pm \sqrt{2}$$

$$\lambda_1 = -1-\sqrt{2}$$

$$-a - 2b = -a - \sqrt{2}a$$

$$2b = \sqrt{2}a$$

$$b = \frac{1}{\sqrt{2}}a$$

$$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1+\sqrt{2}$$

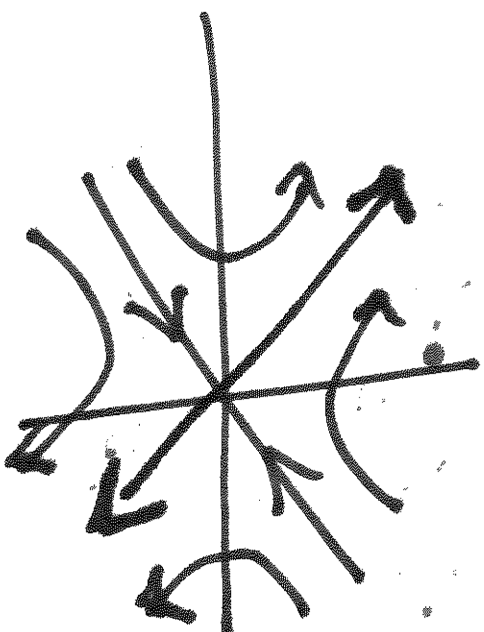
$$-c - 2d = -c + \sqrt{2}c$$

$$-c - d = -d + \sqrt{2}d$$

$$-c = \sqrt{2}d$$

$$-2d = \sqrt{2}c$$

$$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$



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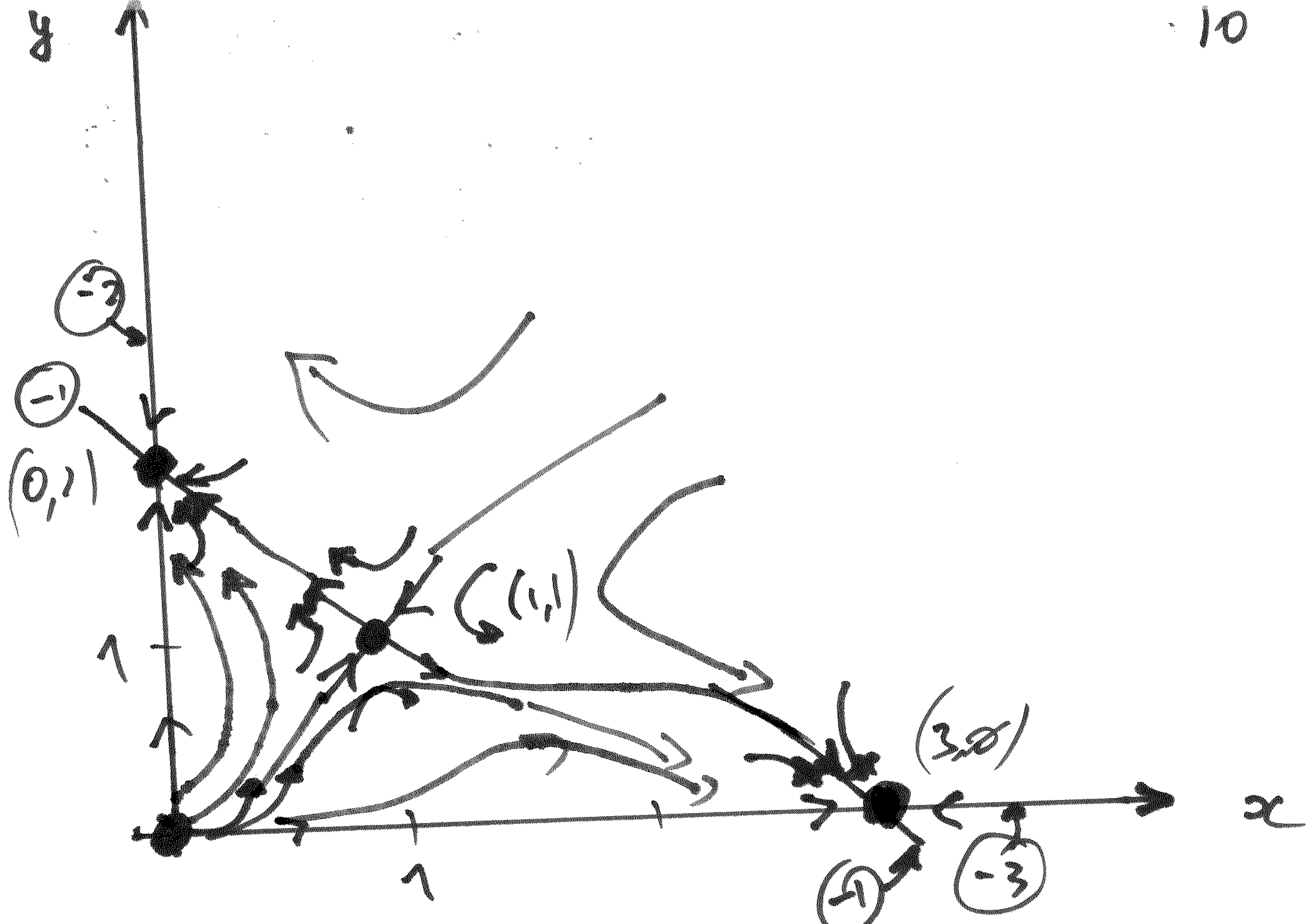
nullcline $x=\emptyset$, $y=\emptyset$

$\dot{x}=\emptyset \Rightarrow x=3-2y$

$\dot{y}=\emptyset \Rightarrow y=2-x$


y
x
-3
-1
(0,1)
1
(1,1)
(3,0)
-1
-3
1

Conservative systems

2nd Newton equation $m\ddot{x}(t) = \underset{\text{force}}{F(x(t))}$

Potential: $F(x) = -\frac{d}{dx}\underbrace{V(x(t))}_{\text{potential}}$

$$m\ddot{x}(t) = -\frac{d}{dx} V(x(t))$$

$$m\ddot{x}(t) + \frac{d}{dx} V(x(t)) = 0$$

$$\underbrace{m\ddot{x}(t)\dot{x}(t)}_{\frac{1}{2}\frac{d}{dt}(\dot{x})^2} + \overbrace{\dot{x}(t)\frac{d}{dx} V(x(t))}^{\frac{d}{dt} V(x(t))} = 0$$

K

$$\frac{d}{dt}\left(\frac{m}{2}(\dot{x})^2 + v(x(t))\right) = 0$$

$$\frac{m}{2}(\dot{x})^2 + v(x(t)) = E$$

$\uparrow$ energy

$$\frac{\partial}{\partial t} E = 0, \quad E = E(x, \dot{x}, t)$$

Energy is conserved

$$\frac{dE}{dt} = 0$$

System $\dot{\underline{x}} = \underline{f}(\underline{x}(t))$ is conservative,
if there exists a quantity
$E = E(\underline{x}, \dot{\underline{x}}, t)$ such that $\frac{d}{dt} E\left(\underline{x}(t), \dot{\underline{x}}(t), t\right) = 0$

Example. Consider $m = 1$, $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$

Find and classify fixed points.

Plot phase portrait.

Solution.

Calculate force: $f(x) = -\frac{d}{dx} V(x)$

$$= -\frac{d}{dx}\left(-\frac{x^2}{2} + \frac{x^4}{4}\right)$$

$$= x - x^3$$

$m\ddot{x} = f(x)$

$\ddot{x} - x + x^3 = \emptyset$

$\dot{x} = y$

$\dot{y} = \ddot{x} = x - x^3$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} y \\ x - x^3 \end{pmatrix}$$

Fixed points: $f(x,y) = y = 0$

$g(x,y) = x - x^3 = 0 \Rightarrow (0,0),\ (\pm 1, 0)$

$$J(x,y) = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$$

$(0,0)$: $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \lambda^2 - 1 = 0,$

$\lambda = \pm 1$

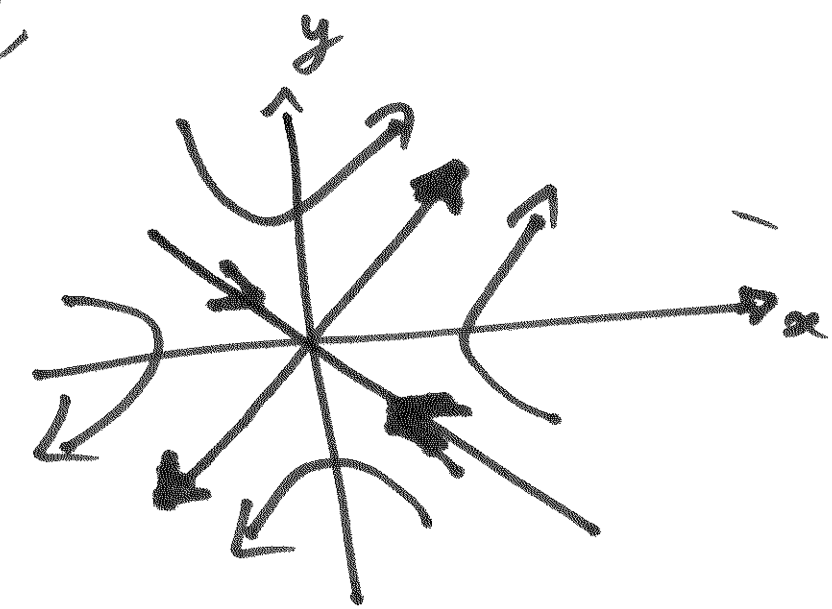
$\underline{\lambda_1 = 1}$: $b = a$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1;$

$b = -a$

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Consider $(1, \emptyset)$

$$J = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$$

$$J(1, \emptyset) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

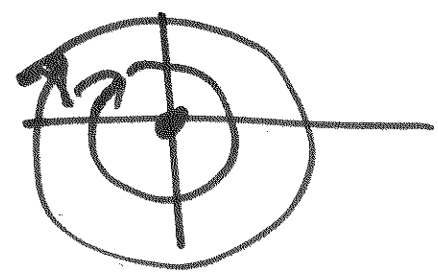
$Tr = \emptyset;$

$Det = 2$ (center)

$$\lambda_{1,2} = \pm\sqrt{2}\, i$$

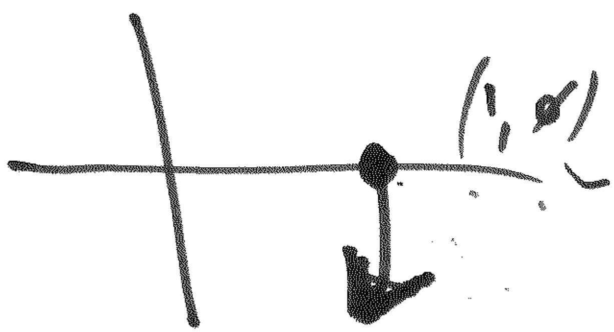
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & \emptyset \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \quad x = 1, y = \emptyset \quad 0\ 1$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$



$x = 2,\ y = 0$

$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$



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