$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\underline{f}(\underline{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

$$\frac{d}{dt}\underline{x}(t) = \underline{f}(\underline{x}(t));$$

$$\frac{d}{dt}x(t) = f_1(x(t), y(t))$$

$$\frac{d}{dt}y(t) = f_2(x(t), y(t))$$

Fixed Points

x^*, y^*

$$f_1(x^*, y^*) = f_2(x^*, y^*) = 0$$

$$x(t) = x^* + \xi(t)$$

$$y(t) = y^* + \nu(t)$$

$$|\xi| \ll 1$$

$$|\nu| \ll 1$$

$$\dot{\eta}(t) = f_1(x^* + \eta(t), y^* + \nu(t))$$

$$\dot{\nu}(t) = f_2(x^* + \eta(t), y^* + \nu(t))$$

$$f_1(x^* + \eta, y^* + \nu) = f_1(x^*, y^*) + \frac{\partial}{\partial x} f_1(x^*, y^*)\eta + \frac{\partial}{\partial y} f_1(x^*, y^*)$$

$$f_1(x^* + \eta, y^* + \nu) = \left(f_1 + \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\Bigg|_{\substack{x = x^* \\ y = y^*}}$$

$$f_2(x^* + \eta, y^* + \nu) = f_2(x^*, y^*) + \frac{\partial}{\partial x} f_2(x^*, y^*)$$

$$+ \frac{\partial}{\partial y} f_2$$

$$f_2(x^* + \eta, y^* + \nu) = f_2(x^*, y^*) +$$

$$+ \frac{\partial}{\partial x} f_2(x^*, y^*) \cdot \eta + \frac{\partial}{\partial y} f_2(x^*, y^*)\nu$$

$$= \left(f_2 + \eta \frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial y} \cdot \nu \right)_{\substack{x = x^* \\ y = y^*}}$$

$$f_1(x^*, y^*) = f_2(x^* y^*) = 0 \qquad \underline{\dot{x}} = \underline{f}(\underline{x})$$

$$\left.\begin{aligned} \dot{\eta}(t) &= \frac{\partial f_1(x^* y^*)}{\partial x} \eta(t) + \frac{\partial f_1(x^*, y^*)}{\partial y} \nu(t) \\ \dot{\nu}(t) &= \frac{\partial f_2(x^* y^*)}{\partial x} \eta(t) + \frac{\partial f_2(x^* y^*)}{\partial y} \nu(t) \end{aligned}\right\}$$

$$\frac{d}{dt}\begin{pmatrix} \eta(t) \\ \nu(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(x^* y^*)}{\partial x} & \frac{\partial f_1(x^* y^*)}{\partial y} \\ \frac{\partial f_2(x^* y^*)}{\partial x} & \frac{\partial f_2(x^* y^*)}{\partial y} \end{pmatrix} \begin{pmatrix} \eta(t) \\ \nu(t) \end{pmatrix}$$

ODE systems with
matrices that can not
be diagonized.

$A \in R^{2 \times 2}$

A has only only one
eigenvector

A is "defective"
matrix

$x(t)$ should have 2
linearly indepentent solutions.

$\underline{\underline{A}} \in R^{2\times 2}$,

λ is an eigenvalue and $\underline{v}$ is an eigenvector $\qquad \underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$; $\qquad \underline{v} e^{\lambda t} = \underline{x}(t)$

$$\underline{x}(t) = (\underline{a} + \underline{v}t)e^{\lambda t}$$

$$\dot{\underline{x}}(t) = (\lambda \cdot \underline{a} + \lambda \underline{v} t + \underline{v})e^{\lambda t}$$

$$\underline{\underline{A}}\,\underline{x}(t) = \underline{\underline{A}}(\underline{a} + \underline{v}t)e^{\lambda t} =$$

$$= (\underline{\underline{A}}\,\underline{a} + \underline{\underline{A}}\,\underline{v}\,t)e^{\lambda t}$$

$$= (\underline{\underline{A}}\,\underline{a} + \lambda \underline{v} t)e^{\lambda t}$$

$$\left(\lambda \cdot \underline{a} + \underline{\lambda \underline{v}} t + \underline{v}\right) e^{\lambda t} = \left(\underline{\underline{A}}\,\underline{a} + \underline{\lambda \underline{v} t}\right) e^{\lambda t}$$

$$\lambda \underline{a} + \underline{v} = \underline{\underline{A}}\,\underline{a}$$

$$\underline{\underline{A}}\,\underline{a} - \lambda \underline{a} = \underline{v}$$

$$\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) \underline{a} = \underline{v}$$

Find $\underline{a}$

$$x(t) = C_1 e^{\lambda t} \cdot \underline{v} +$$

$$+ C_2 e^{\lambda t} \left(\underline{a} + \underline{v} t\right)$$

$$\left.\begin{aligned} \dot{x} &= x - y \\ \dot{y} &= 4x - 3y \end{aligned}\right\}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(1-\lambda)(-3-\lambda) + 4 = 0$$

$$(\lambda-1)(\lambda+3) + 4 = 0$$

$$\lambda^2 + 2\lambda - 3 + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0 \quad \Rightarrow \lambda = -1$$

$$\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -1\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a - b = -a \qquad \lambda = -1$$

$$2a = b,$$

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (A - \lambda \underline{I})\underline{a} = \underline{v}$$

$$A - \lambda \underline{I} = A + I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2a_1 - a_2 = 1 \\ 4a_1 - 2a_2 = 2 \end{cases}$$

$$2a_1 - a_2 = 1$$

$$a_1 = 1, \quad a_2 = 2a_1 - 1 = 1$$

$$\underline{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) e^{-t}$$

$$\underline{\dot{x}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \underline{x}, \quad \lambda > \emptyset;$$

Eigenvalue $\lambda = \lambda$

Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{a} = \underline{v}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$a_2 = 1$$
$$\emptyset \cdot a_1 + \emptyset a_2 = \emptyset \quad \Rightarrow \underline{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{x}(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} + C_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right) e^{\lambda t}$$

$$A = \begin{pmatrix} -2 & 1 \\ 0 & a \end{pmatrix}$$

a is a given parameter

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalue

$$(-2-\lambda)(a-\lambda) - 0 = 0$$

$$(\lambda+2)(\lambda-a) - 0 = 0$$

$$\lambda = -2$$

$$\lambda = a$$

$$A = \begin{pmatrix} -2 & 1 \\ 0 & a \end{pmatrix}$$

$$\underline{v}_1 \Rightarrow \begin{pmatrix} -2 & 1 \\ \emptyset & a \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = -2 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$-2 v_{11} + v_{12} = -2 v_{11}$$

$$\left.\begin{matrix} v_{12} = \emptyset \\ v_{11} = 1 \end{matrix}\right\} = \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{v}_2 \Rightarrow -2 v_{21} + v_{22} = a v_{21}$$

$$= a v_{21}$$

$$(-2-a) v_{21} + v_{22} = \emptyset$$

$$v_{22} = (2+a) v_{21}$$

$$\underline{v}_2 = \begin{pmatrix} 1 \\ 2+a \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \emptyset & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

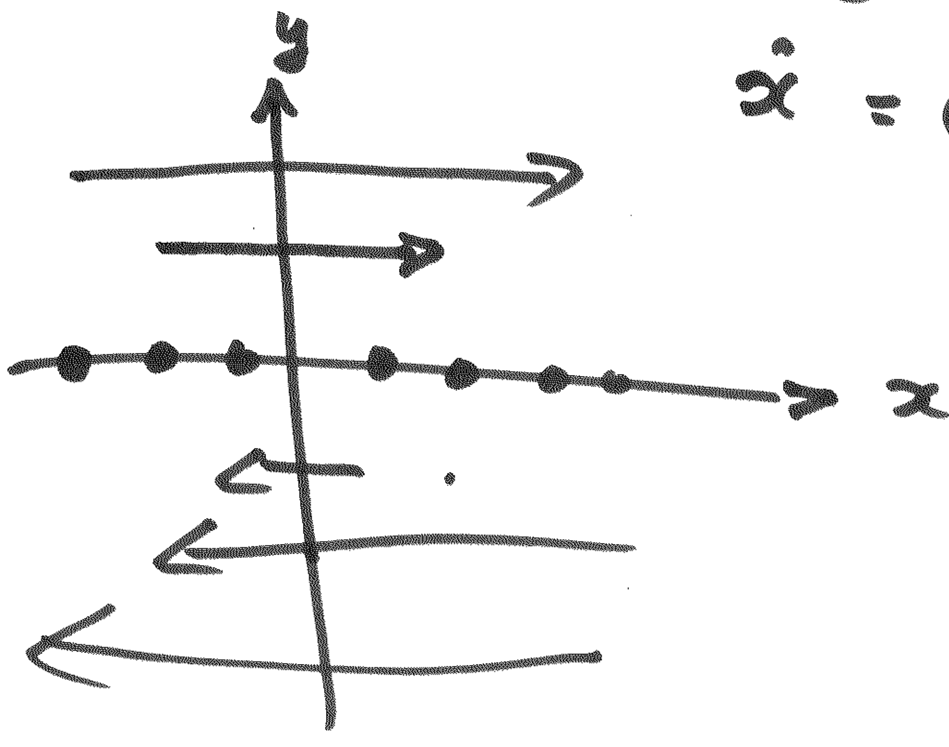
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 2+a \end{pmatrix} e^{at}$$

$$\underline{\dot{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{x}$$

$$\lambda_1 = \lambda_2 = \emptyset \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \emptyset \end{pmatrix}$$

$$\dot{y} = \emptyset \Rightarrow y = C_1;$$

$$\dot{x} = C_1 \Rightarrow x = C_1 t + C_2$$



$y = \emptyset \Rightarrow$ nonisolated fixed points

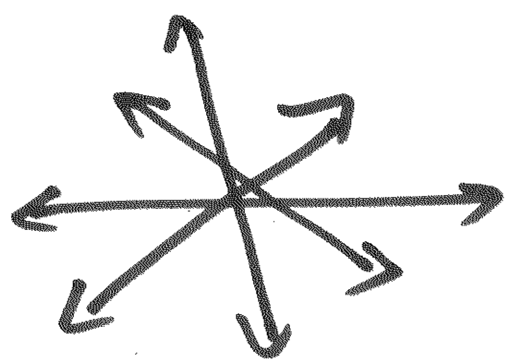
$$\underline{\dot{x}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \underline{y} \Rightarrow \dot{x} = \dot{y} = \emptyset$$

Any point on x, y is a fixed point

One eigenvalue and 2 distinct eigenvectors.

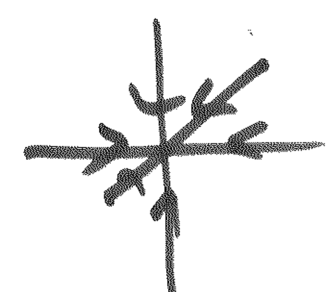
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x(t) = x_0 e^{t} \\ y(t) = y_0 e^{t} \end{array}$$

$$y(t) = \frac{y_0}{x_0}\, x(t)$$



unstable star

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$$



Stable star

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4Det(A)}}{2}$$

$$Det(A) = \frac{1}{4} Tr^2(A)$$

- $Det\ A < \emptyset$: <u>Area 4</u>

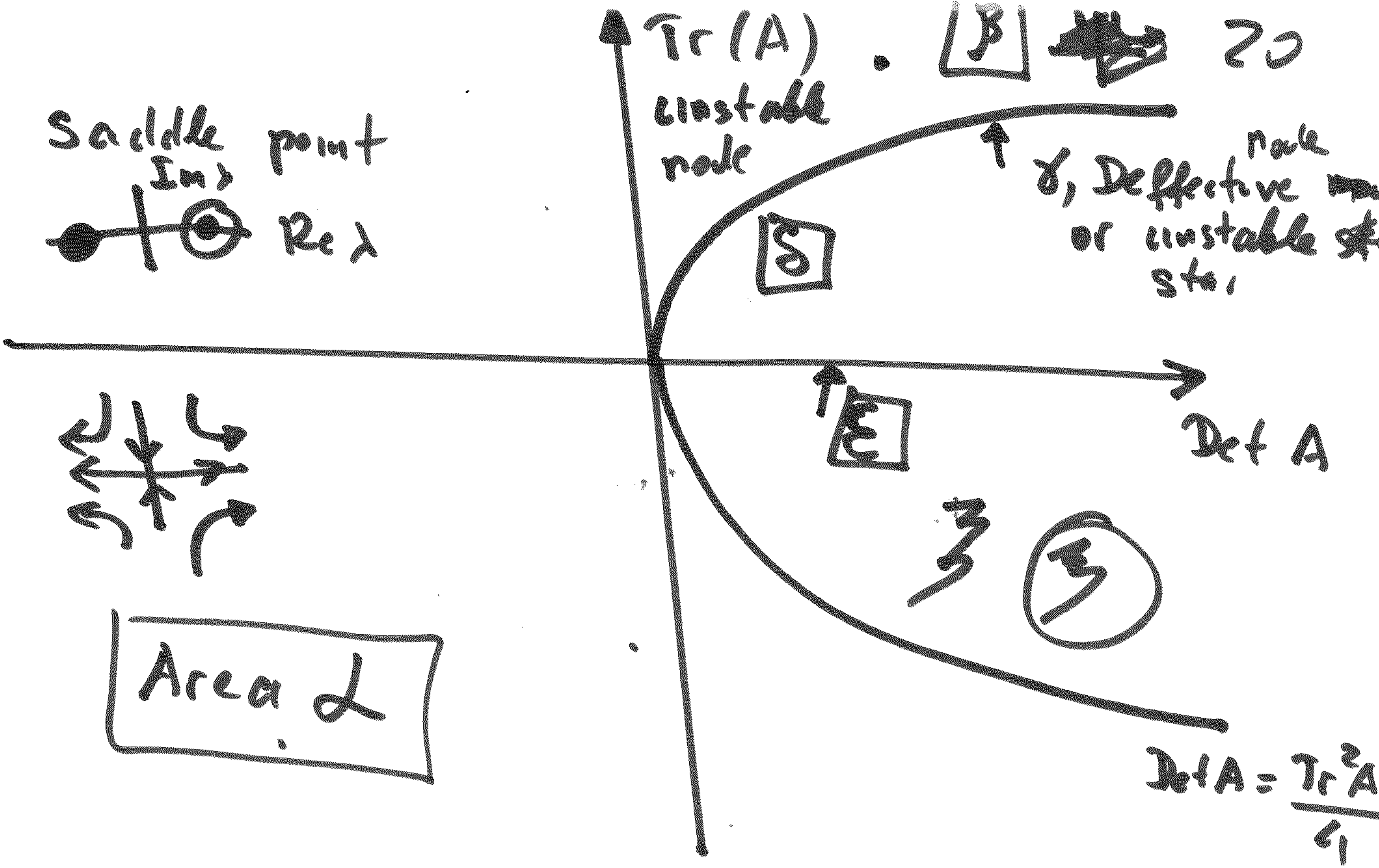
$Tr^2(A) - 4 Det A > \emptyset$

$\sqrt{Tr^2(A) - 4 Det A} > Tr(A)$

Saddle point

$\lambda_1 \cdot \lambda_2 < \emptyset$

Two real roots opposite sign



Tr (A)
β
unstable node
γ, Deffective node
or unstable star
δ
ε
ζ
Det A
Det A = $\frac{Tr^2 A}{4}$
Saddle point
Im λ
Re λ
Area α

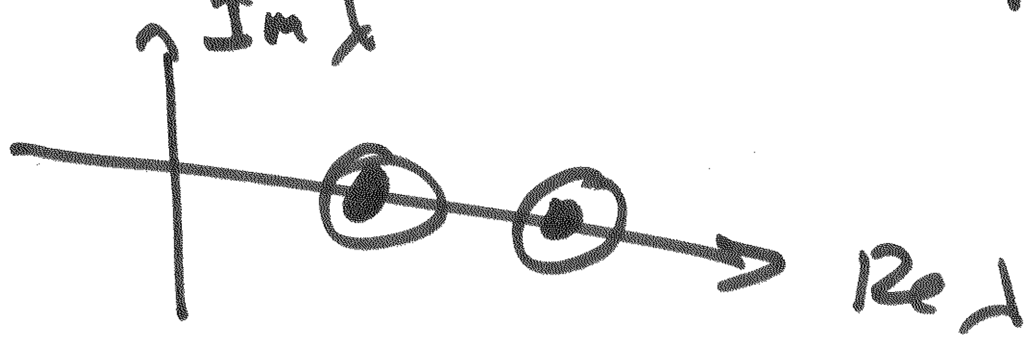
- $\text{Det}(A) > \emptyset$, $\text{Tr}(A) > \emptyset$, $\text{Tr}^2(A) > 4\,\text{Det}$

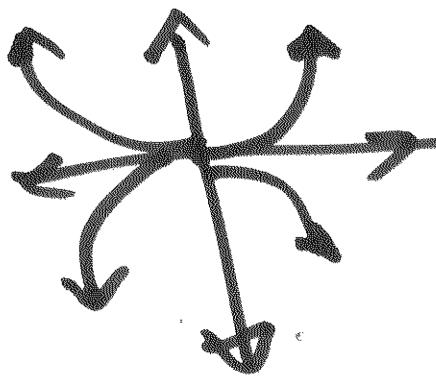
Area)3

$\text{Tr}\,A > \sqrt{\text{Tr}^2(A) - 4\,\text{Det}(A)} > \emptyset$

2 positive roots, both real

$\lambda_1 > \lambda_2 > \emptyset$ unstable node

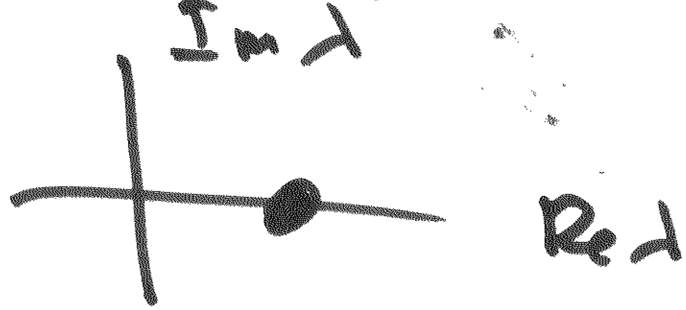




- Area γ $\Delta > \emptyset$, $\text{Det } A > \emptyset$, $\text{Tr}(A) > \emptyset$

  $\text{Tr}^2 = 4 \text{ Det}$

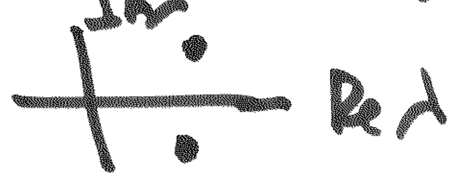
  $\lambda_1 = \lambda_2$ real positive #

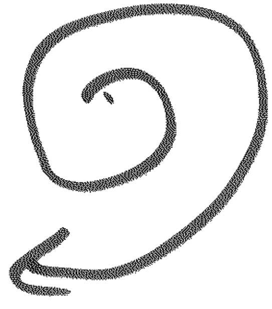


Deffective node, unstable star

- Area Δ: $\text{Det } A > \emptyset$, $\text{Tr} > \emptyset$, $\text{Tr}^2(A) < 4 \text{ Det}(A)$

  2 complex conjugate real roots with positive real part





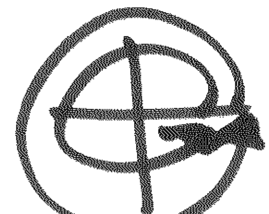
unstable spiral

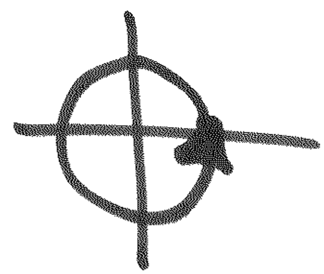
Area $\mathcal{E}$:

$\mathrm{Det}\, A > 0$, $\mathrm{Tr}\, A = 0$,

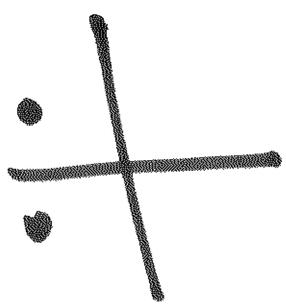
2 complex conjugate roots with zero real part.

center





~~Area~~ Area Ξ



$\mathrm{Det} > 0$

$\mathrm{Tr} < 0$

$\mathrm{Tr}^2 < 4\,\mathrm{Det}$

2 complex conjugate roots with negative real part

stable Spiral