$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad \frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t) =$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underline{x}(t)$$

Eigenvalues of $\underline{\underline{A}}$: λ_1, λ_2

$\underline{x}_1 ; \underline{x}_2$

$$\underline{\underline{A}}\,\underline{x}_1 = \lambda_1 \underline{x}_1$$

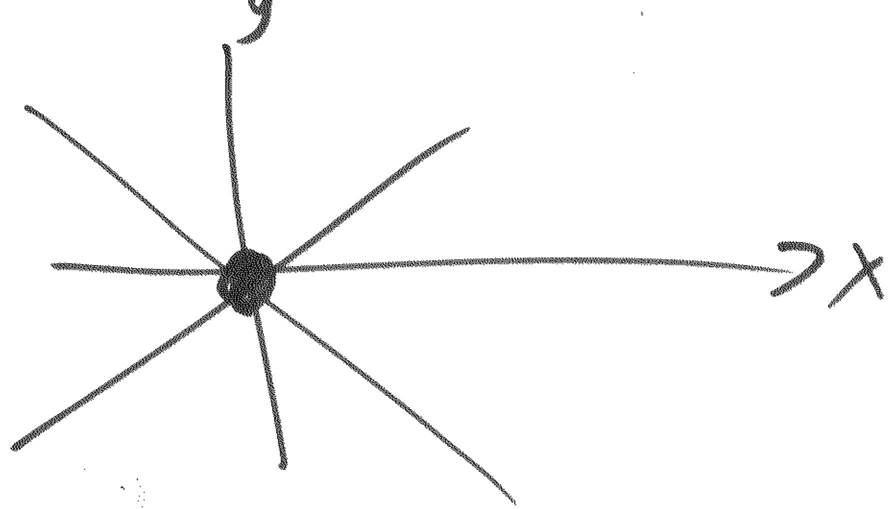
$$\underline{\underline{A}}\,\underline{x}_2 = \lambda_2 \underline{x}_2$$

$$\mathrm{Det}(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \emptyset$$

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x};$$

$$(\underline{\underline{A}} - \lambda I)\,\underline{x} = \emptyset$$

$$\left.\begin{array}{l} ax + by = \emptyset \\ cx + dy = \emptyset \end{array}\right\} \Rightarrow \quad \begin{array}{l} y = -\dfrac{ax}{b} \\ y = -\dfrac{cx}{d} \end{array}$$



$$\frac{a}{b} = \frac{c}{d}$$

Lines
coincide

$$ad - cb = \emptyset$$

$$\mathrm{Det}(A) = ad - cb$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \emptyset;$$

~~a+d = Rat~~

$\boxed{a+d = Tr(A)}$

$\boxed{D = ad - bc}$

$$(a-\lambda)(d-\lambda) - bc = \emptyset$$

$$(\lambda-a)(\lambda-d) - bc = \emptyset$$

$$\lambda^2 - (a+d)\lambda + ad - bc = \emptyset$$

root

$$\lambda^2 + Tr(A)\lambda + Det(A) = \emptyset$$

$$\lambda_{1,2} = \frac{+Tr(A) \pm \sqrt{Tr^2(A) - 4Det(A)}}{2}$$

$Tr^2(A)$ relates to $4Det$

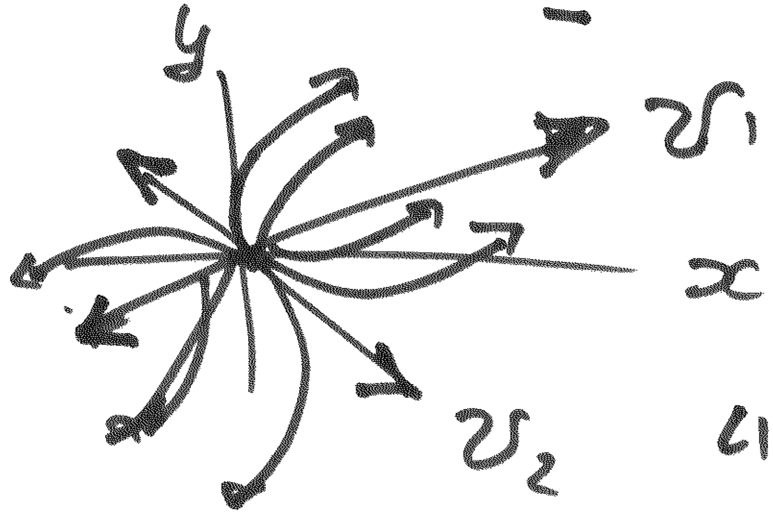
$\boxed{\text{a, b, c, d are real \#}}$

- $Tr^2(A) > 4Det$
  2 real distinct roots
- $Tr^2(A) < 4Det$
  2 complex roots that are complex conjugate of each other
- $Tr^2 A = 4Det$, one real double

$Tr^2 A > 4\,Det$

$\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$

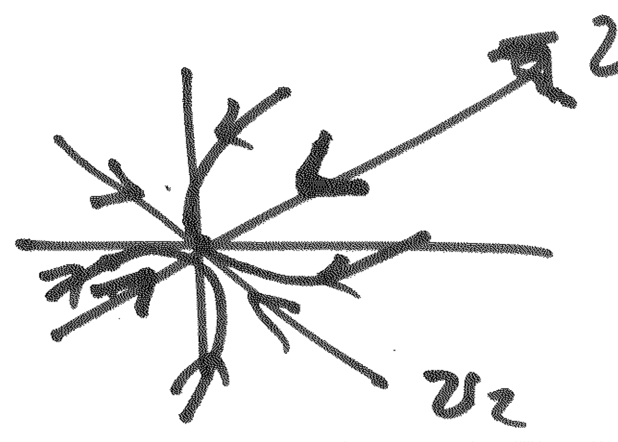
- $\lambda_1 > \lambda_2 > \emptyset$



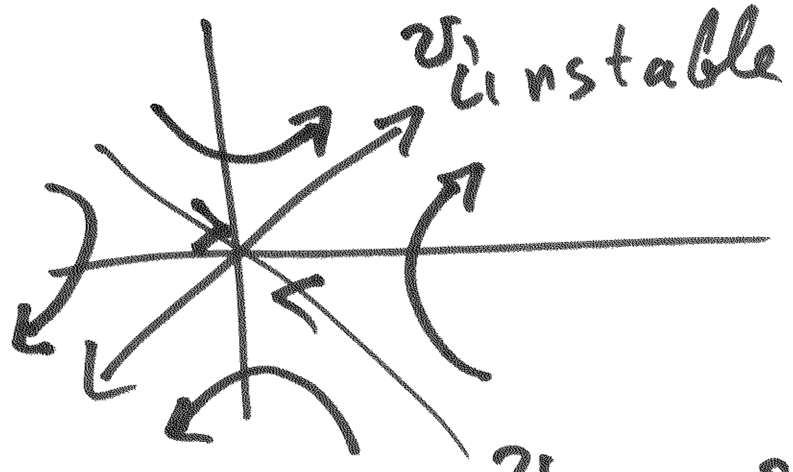
unstable node

Trajectories are tangential to v_2

- $\lambda_1 < \lambda_2 < \emptyset$



- $\lambda_2 < 0 < \lambda_1$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix}$$

$$\dot{x} = x + y$$
$$\dot{y} = 4x - 2y$$

$$(x_0, y_0) = (2, -3) = (2, -3)$$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}; \quad Tr = -1$$

$$Det = -6$$

$$\lambda_{12} = \frac{+Tr \pm \sqrt{Tr^2 - 4D}}{2}$$

$$= \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm 5}{2} = -3;\ 2$$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}; \quad 4a - 2b = 2b$$

$$4a = 4b \Rightarrow a = b$$

$$a + b = 2a; \quad a = b \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -3\begin{pmatrix} c \\ d \end{pmatrix}$$

$$c + d = -3c; \quad 4c + d = 0$$

$$d = -4c$$

$$v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$\lambda_1 = 2; \; v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3; \; v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

Initial conditions

$$\underline{x}(t=0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

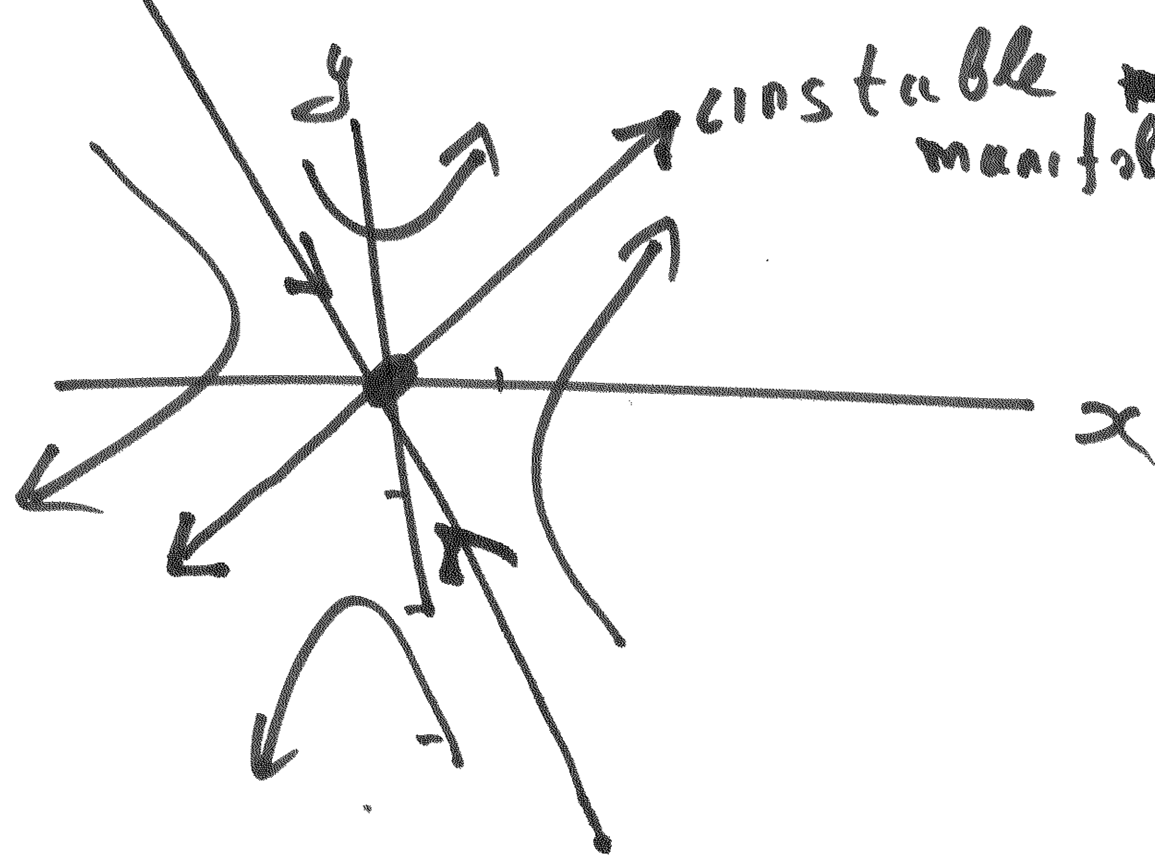
$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ c_1 - 4c_2 &= -3 \end{aligned}\right] \quad \begin{aligned} -c_1 + 4c_2 &= 3 \\ c_1 + c_2 &= 2 \end{aligned}$$

$5c_2 = 5, \; c_2 = 1 = c_1$

$$x(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

Plot Phase Portrait for $\dot{x} = x+y$

$\dot{y} = 4x - 2y$



$$e^{ix} = \cos x + i \sin x$$

$$e^{x} = 1 + x + x^2/2 + x^3/6 + \dots \qquad (i)^{2m} = \left((i)^2\right)^m = (-1)^m$$

$$e^{x} = \sum_{k=0}^{\infty} x^k / k!$$

$$f(x) = \sum_{k=0}^{\infty} \left(\left(\frac{d}{dx}\right)^k f(x) \Big|_{x=0} \right) \frac{x^k}{k!}$$

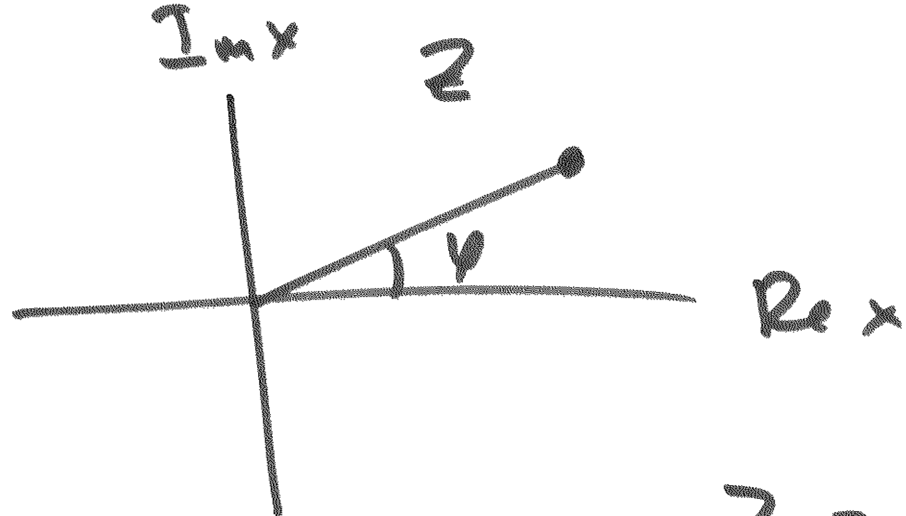
$$e^{ix} = \sum_{k=0}^{\infty} x^k (i)^k / k! =$$

$k = 2m,\ m = 0..\infty$

$k = 2n+1,\ n = 0..\infty$

$$= \sum_{m=0}^{\infty} x^{2m} (i)^{2m} / (2m)! + \sum_{n=0}^{\infty} \frac{x^{2n+1} (i)^{2n} i}{(2n+1)!}$$

$$e^{ix} = \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2m!}}_{\cos x} + i \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin x}$$



$$\boxed{\begin{aligned} R^2 &= x^2+y^2 \\ \tan\varphi &= y/x \end{aligned}}$$

$$\cancel{Z = x + i\cos}$$

$$Z = x + iy = R e^{i\varphi}$$

$$\lambda_{2} = \frac{Tr(A) \pm \sqrt{Tr^2 A - 4\,Det}}{2}$$

$$4\,Det > Tr^2 A$$

$$\lambda_{1,2} = \gamma \pm i\omega; \qquad \lambda_1 = \gamma + i\omega \quad \underline{v}_1 = a + ib$$

$$\lambda_2 = \gamma - i\omega \quad \underline{v}_2 = a - ib$$

$$\gamma = Re\,\lambda = \frac{Tr(A)}{2}$$

$$\omega = Im\,\lambda = \frac{1}{2}\sqrt{4\,Det\,A - Tr^2 A}$$

$$\underline{v}_1 = \underline{a} + i\underline{b} \qquad \underline{a} = Re\ v$$

$$\underline{v}_2 = \underline{a} - i\underline{b} \qquad \underline{b} = Im\ \underline{v}$$

$$\underline{\lambda} = i\omega + \gamma$$

$$\underline{v} = \underline{a} + i\underline{b}$$

$$\underline{\underline{A}}\,(\underline{a} + i\underline{b}) = (\gamma + i\omega)(\underline{a} + i\underline{b}), \qquad \underline{\underline{A}}^{*} = \underline{\underline{A}}$$

$$\underline{\underline{A}}\,(a - ib) = (\gamma - i\omega)(\underline{a} - i\underline{b})$$

$$\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} \qquad (1)$$

$$= c_1 (\underline{a} + i\underline{b}) e^{(\gamma + i\omega)t} \qquad (2)$$

$$+ c_2 (\underline{a} - i\underline{b}) e^{(\gamma - i\omega)t} \qquad (3)$$

$$= c_1 (\underline{a} + i\underline{b}) e^{\gamma t} (\cos\omega t + i \sin\omega t) \qquad (4)$$

$$+ c_2 (\underline{a} - i\underline{b}) e^{\gamma t} (\cos\omega t - i \sin\omega t) \qquad (5)$$

$$= c_1 (\underline{a} \cos\omega t - \underline{b} \sin\omega t) e^{\gamma t} \qquad (6)$$

$$+ c_2 (\underline{a} \cos\omega t - \underline{b} \sin\omega t) e^{\gamma t} \qquad (7)$$

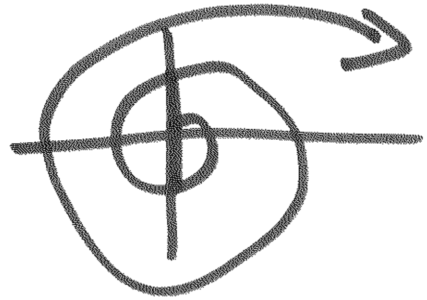
$$+ i c_1 (\underline{a} \sin\omega t + \underline{b} \cos\omega t) e^{\gamma t} \qquad (8)$$

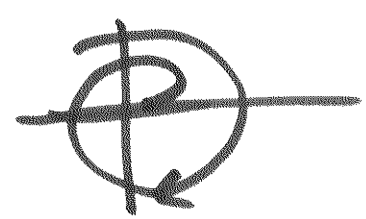
$$+ i c_2 (-\underline{a} \sin\omega t - \underline{b} \cos\omega t) e^{\gamma t} \qquad (9)$$

$$\underline{x}(t) = (C_1 + C_2)(\underline{a}\cos\omega t - \underline{b}\sin\omega t)e^{\gamma t}$$
$$+ i(C_1 - C_2)(\underline{a}\sin\omega t + \underline{b}\cos\omega t)e^{\gamma t}$$

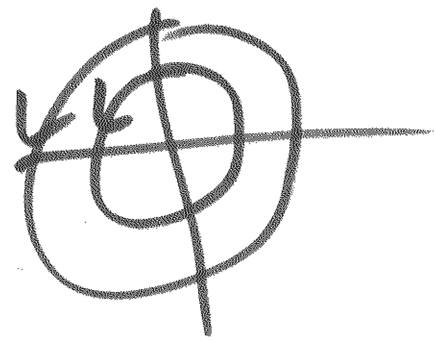
$C_1 + C_2 = D_1$; $i(C_1 - C_2) = D_2$

$$\boxed{\begin{aligned} x(t) = {} & D_1(\underline{a}\cos\omega t - \underline{b}\sin\omega t)e^{\gamma t} \\ & + D_2(\underline{a}\sin\omega t + \underline{b}\cos\omega t)e^{\gamma t} \end{aligned}}$$

$\gamma > 0$  unstable spiral

$\gamma < 0$  stable spiral

$r = \emptyset$ - center



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$$\underline{\dot{x}}(t) = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \underline{x}(t); \quad x(t=0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\mathrm{Tr}\,A = 3-3=0; \quad \mathrm{Det}\,A = -9+36 = 27$$

$$\lambda_{1,2} = \frac{\pm\sqrt{27}}{2} \cdot i \cdot 2 = i\sqrt{27} = 3\sqrt{3}\,i$$

$$\underline{v}_1 = \underline{a} + \underline{b} = \begin{pmatrix} k+il \\ m+in \end{pmatrix} = \begin{pmatrix} k \\ m \end{pmatrix} + i \begin{pmatrix} l \\ n \end{pmatrix}$$

$$3(k+il) + 9(m+in) = 3\sqrt{3}i\,(k+il)$$

$$k+3m = -\sqrt{3}\,l$$

$$l+3n = \sqrt{3}\,k \qquad v_{1,2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

$$l=0, \quad k=3m, \quad \sqrt{3}n = k$$

21)

$$\lambda_1 = \sqrt{3}\cdot 3i; \quad v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

$$\lambda_2 = -3\sqrt{3}i; \quad v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

$$x(t) = D_1 \left[\begin{pmatrix} 3 \\ -1 \end{pmatrix} \cos 3\sqrt{3}t - \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \sin 3\sqrt{3}t \right]$$

$$+ D_2 \left[\begin{pmatrix} 3 \\ -1 \end{pmatrix} \sin 3\sqrt{3}t + \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \cos 3\sqrt{3}t \right]$$

$$x(0) = D_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + D_2 \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} =$$

$$= \begin{bmatrix} 3D_1 \\ -D_1 + \sqrt{3}D_2 \end{bmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad \begin{array}{l} D_1 = 2/3 \\ \sqrt{3}D_2 = -4 + 2/3 = -10/3 \end{array}$$

$D_1 = 2/3$

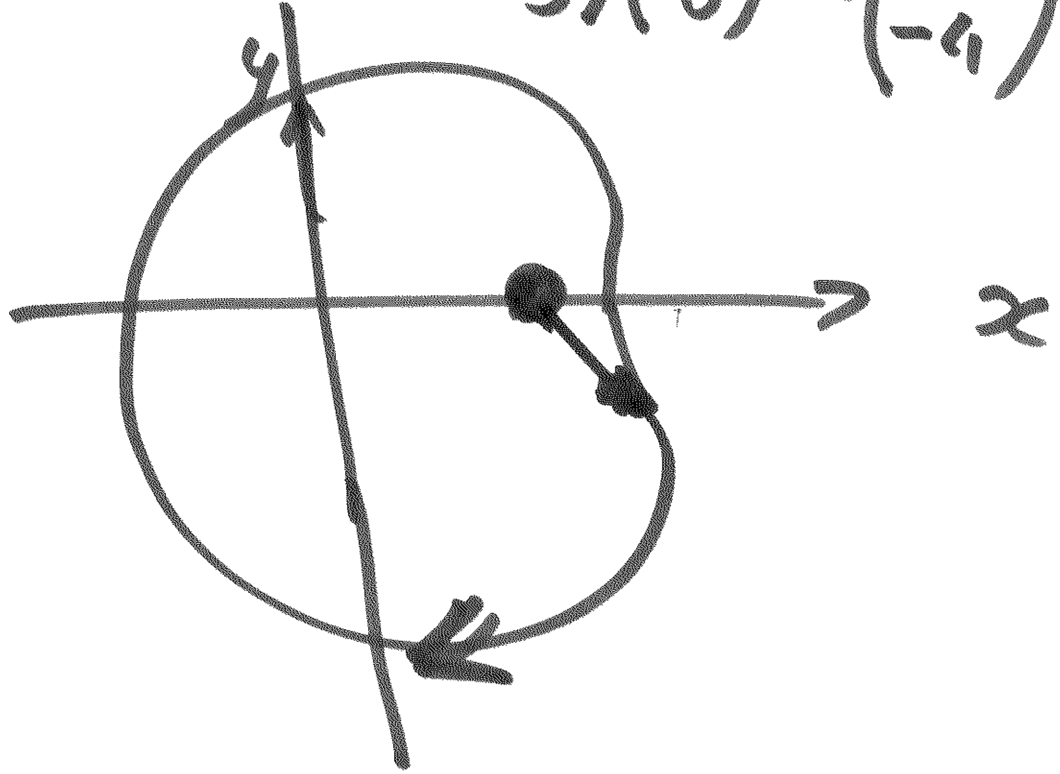
$D_2 = -\frac{10}{3\sqrt{3}}$

$$x(t) = \frac{2}{3}\left[\begin{pmatrix}3\\-1\end{pmatrix}\cos 3\sqrt{3}t - \begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\sin 3\sqrt{3}t\right]$$
$$- \frac{10}{3\sqrt{3}}\left[\begin{pmatrix}3\\-1\end{pmatrix}\sin 3\sqrt{3}t + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix}\cos 3\sqrt{3}t\right]$$

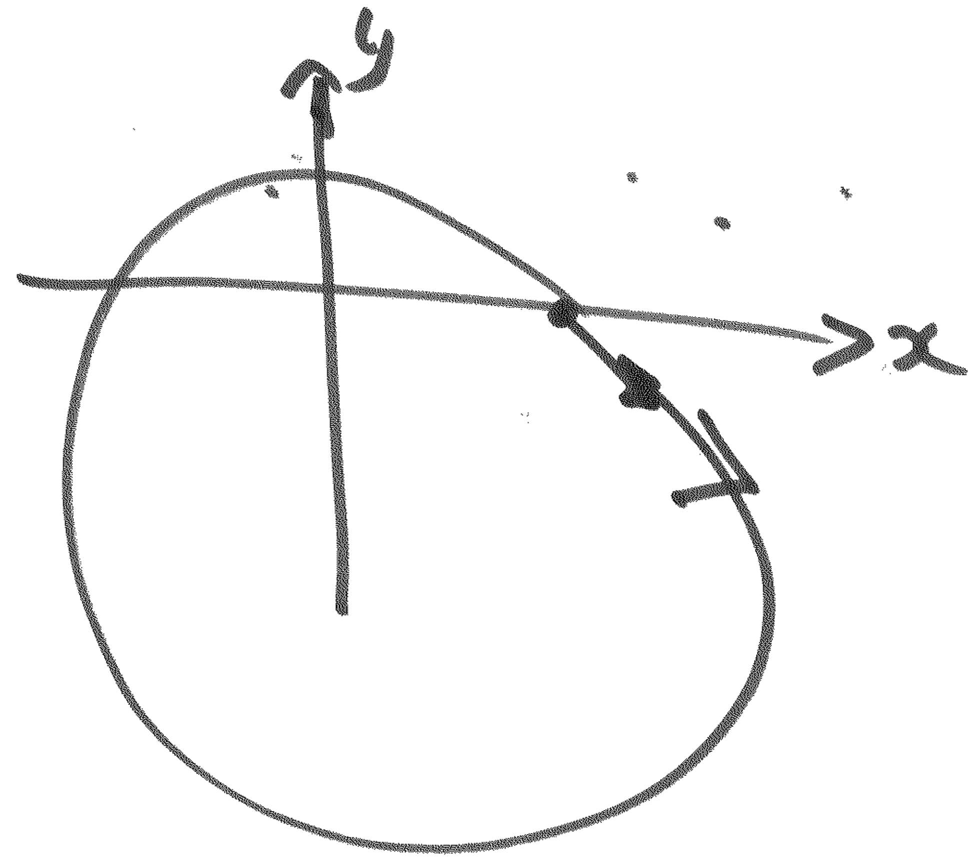
$$\underline{\dot{x}} = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \underline{x}$$

$$\underline{x}(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{\dot{x}}(t=0) = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$



$\underline{x}(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$$\dot{\underline{x}} = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \underline{x}; \quad x(t=0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix};$$

$$Tr = 4; \; Det = 3 + 65 = 68$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 68}}{2}$$

$$= 2 \pm \sqrt{4 - 68} = 2 \pm 8i$$

$$v_1 = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$