$$\begin{pmatrix} \vdots \\ x_n(t) \end{pmatrix} ; \quad \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}$$

$$\frac{d}{dt} \underline{x}(t) = \underline{\underline{A}}\, \underline{x}(t); \quad x(t=0) = D$$

$$\frac{d}{dt} \underline{x}(t) = \epsilon \underline{\underline{A}}\, \underline{x}_0(t);$$

$$x(t) = \sum_{k=0}^{\infty} \frac{\underline{\underline{A}}^k t^k}{k!} \underline{x}_0 \cdot \epsilon^k$$

$\underline{\epsilon = 1}$

$$\underline{x}(t) = \sum_{k=0}^{\infty} \frac{\underline{\underline{A}}^k \cdot t^k}{k!} \underline{z}_0 = \underline{z} + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!} \underline{z}_0$$

$$\frac{d}{dt} \underline{x}(t) = \sum_{k=0}^{\infty} \frac{\underline{\underline{A}}^k}{k!} \cdot \underline{z}_0 \frac{d}{dt}(t^k)$$

$$A^k = A^{k-1} \cdot A = A \cdot A^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{\underline{\underline{A}}^k}{k!} \cdot \underline{z}_0 \cdot k \cdot t^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{A^k}{k!} \underline{z}_0 \cdot k \cdot t^{k-1} =$$

$$= \sum_{k=1}^{\infty} A \frac{A^{k-1}}{(k-1)!} t^{k-1} \underline{z}_0 =$$

$$= A \sum_{k=1}^{\infty} \frac{A^{k-1}}{(k-1)!} t^{k-1} \underline{z}_0 = A \underbrace{\boxed{\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} z_0}}_{x(t)}$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\ddot{\underline{x}} = \frac{d}{dt}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\frac{d}{dt}x = A^2 x$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \equiv e^{At}$$

4

$$\underline{\underline{A}} = (a_{ij}),\ i = 1..n,\ j = 1..n$$

$$a_{ij} = d\,\delta^i_j = d_i \cdot \delta^i_j = \begin{cases} d_i, & i = j \\ \emptyset, & i \neq j \end{cases}$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_n(t) \end{pmatrix};$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{x}(t = \emptyset) = \underline{z}_0 = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$x_i(t) = z_i\, e^{d_i t}$$

$$\underline{x}(t) = \left\{ x_i(t=0)\, e^{d_i t},\ i = 1..n \right\}$$

$$\underline{x}(t) = \sum_{k=0}^{\infty} \underline{\underline{A}}^k\, t^k\, \frac{1}{k!}\, \underline{z}_0$$

$$\underline{\underline{A}} = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & d_3 \ddots & \\ 0 & & & d_n \end{bmatrix}; \quad \underline{\underline{A}}^2 = \begin{bmatrix} d_1^2 & & & 0 \\ & d_2^2 & & \\ & & \ddots & \\ 0 & & & d_n^2 \end{bmatrix}$$

$$\underline{\underline{A}}^k = \begin{pmatrix} d_1^k & & & 0 \\ & d_2^k & & \\ & & \ddots & \\ 0 & & & d_n^k \end{pmatrix}$$

6

$$\underline{x}(t) = \sum_{k=0}^{\infty} \underline{\underline{A}}^k t^k \frac{1}{k!} z_0 =$$

$$= \begin{pmatrix} \sum_0^{\infty} d_1^k t^k / k! & & & 0 \\ & \sum \frac{d_2^k t^k}{k!} & & \\ & & \sum_{k=0}^{\infty} \frac{d_3^k t^k}{k!} & \\ 0 & & & \ddots \\ & & & & \sum_{k=0}^{\infty} \frac{d_n^k t^k}{k!} \end{pmatrix} z$$

$$= \begin{pmatrix} e^{d_1 t} & & & 0 \\ & e^{d_2 t} & & \\ & & e^{d_3 t} & \\ 0 & & & \ddots \\ & & & & e^{d_n t} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$\underline{x}(t) = \begin{pmatrix} e^{d_1 t} z_1 \\ e^{d_2 t} z_2 \\ \vdots \\ e^{d_n t} z_n \end{pmatrix}$$

Take A, calculate eigenvalues:

$$\det(A - \lambda I) = \emptyset;$$

For each eigenvalue calculate eigenvector

$$\underline{\underline{A}}\,\underline{x} = \lambda_i \underline{x};$$

$$\underline{\underline{A}} = V \cdot D \cdot V^{-1};$$

D is a diagonal matrix of eigenvalues

V is a matrix of eigenvectors.

~~Atr~~

$$\underline{\underline{A}} = V \cdot D \cdot V^{-1}$$

$$\underline{\underline{A}}^2 = V \cdot D \cdot V^{-1} \cdot V \cdot D \cdot V^{-1} =$$

$$= V\, D \cdot D \cdot V^{-1} = V D^2 V^{-1}$$

$$\underline{\underline{A}}^k = V \cdot D^k \cdot V^{-1}$$

$$\dot{\underline{x}} = \underline{\underline{A}} \cdot \underline{x}$$

$$\underline{x}(t=0) = \underline{z}_0 = \underline{z}_0 \qquad \underline{x}(t) = \sum_{k=0}^{\infty} \frac{\underline{\underline{A}}^k t^k}{k!} \underline{z};$$

$$\underline{x} = \underline{\underline{V}} \cdot \underline{y} \qquad \overbrace{\underline{\underline{V}} \cdot \dot{\underline{y}}}^{\dot{x}} = \overbrace{V \cdot D \cdot V^{-1}}^{A} \, \overbrace{\underline{\underline{V}} \cdot y}^{x}$$

$$V^{-1} \left[\underline{\underline{V}} \cdot \dot{\underline{y}} = \underline{\underline{V}} \, \underline{\underline{D}} \cdot \underline{y} \right]$$

$$\dot{\underline{y}} = \underline{\underline{D}} \cdot \underline{y} \Rightarrow \underline{y}(t) = \begin{pmatrix} y_{10} e^{d_1 t} \\ y_{20} e^{d_2 t} \\ \vdots \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

$$A = \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix}^{-1}$$

$$\underline{x}(t) = \sum_{k=0}^{\infty} \frac{A^k t^k \underline{z}}{k!} =$$

$$\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{D}}\,\underline{\underline{V}}^{-1}$$

$$= V\left(\sum_{k=0}^{\infty} \frac{D^k t^k}{k!}\right)\left(V^{-1}\underline{z}\right)$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}; \quad \underline{x}(t=0) = \underline{z}$$

$$\underline{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{D}}\,\underline{\underline{V}}^{-1}; \quad x = Vy$$

$$\dot{x} = V\dot{y} = (VDV^{-1})(Vy) = \underline{\underline{V}}\,\underline{\underline{D}}\,\underline{y}$$

$$\dot{\underline{y}} = \underline{\underline{D}}\,\underline{y}; \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$y(t) = \begin{pmatrix} y_{10} e^{\lambda_1 t} \\ y_{20} e^{\lambda_2 t} \end{pmatrix} =$$

$$\underline{y}(t) = \begin{pmatrix} y_{10} e^{\lambda_1 t} \\ y_{20} e^{\lambda_2 t} \end{pmatrix}$$

$$\underline{x} = \underline{\underline{V}}\,\underline{y} = \begin{pmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{pmatrix} \begin{pmatrix} y_{10} e^{\lambda_1 t} \\ y_{20} e^{\lambda_2 t} \end{pmatrix} =$$

$$= \begin{pmatrix} V_{11} y_{10} e^{\lambda_1 t} + V_{21} y_{20} e^{\lambda_2 t} \\ V_{12} y_{10} e^{\lambda_1 t} + V_{22} y_{20} e^{\lambda_2 t} \end{pmatrix}$$

$$= y_{10} e^{\lambda_1 t} \cdot \underline{V_1} + y_{20} e^{\lambda_2 t} \underline{V_2}$$

Let $A \in \mathbb{R}^{n \times n}$,

with n different real eigenvalues
distinct

$\lambda_1, \lambda_2 \ldots \lambda_n$

and corresponding eigenvectors $v_1, v_2 \ldots v_n$.

Then, if all λ's are real,

solution to $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ is given by

$$x(t) = \sum_{k=1}^{n} c_k \, \underline{v}_k \, e^{\lambda_k t}$$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix};$$

$$\underline{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

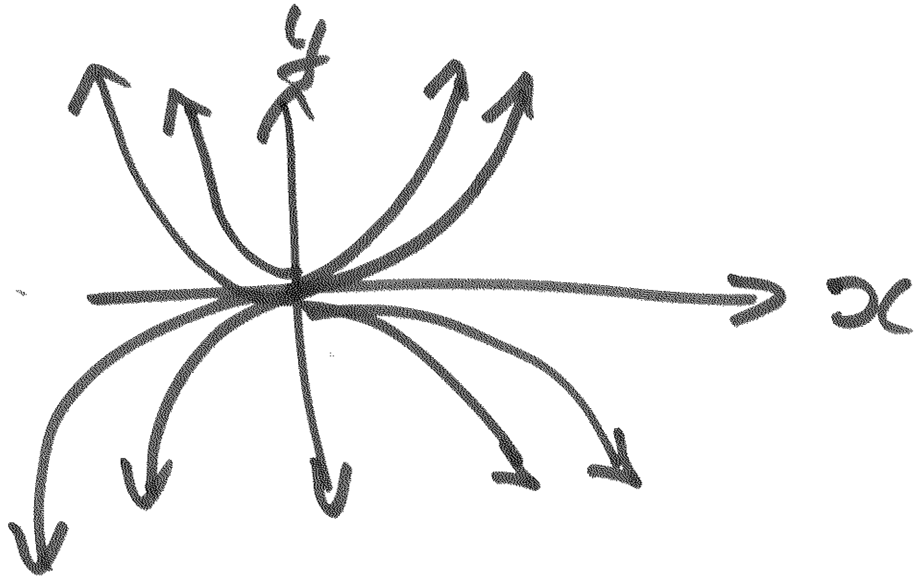
Solve $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$;

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ 2y(t) \end{pmatrix}$$

$x(t) = x_0 e^{t}$

$y(t) = y_0 e^{2t} \Rightarrow$

$e^{t} = x(t)/x_0$; $\quad y(t) = y_0 \dfrac{x^2(t)}{x_0^2}$



unstable node

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues are $\lambda_1 = 1;\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2;\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

Example.

Consider $\underline{\underline{A}} = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}$

Solve $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ and plot phase potraits for all values of λ.

$\underline{x}(t) = x_0 e^{\alpha t}$

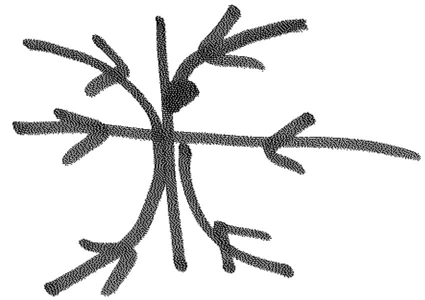
$\underline{y}(t) = y_0 e^{-t}$

$\lambda_1 = \alpha;\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \lambda_2 = -1;\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underline{x}(t) = c_1 \underline{v}_1 e^{\alpha t} + c_2 \underline{v}_2 e^{-t} =$

$= \begin{pmatrix} c_1 e^{\alpha t} \\ c_2 e^{-t} \end{pmatrix}$

$\alpha < -1$



Stable node

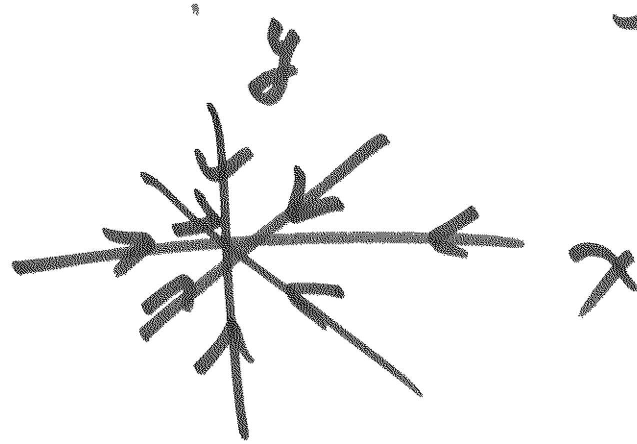
Trajectories are tangential to eigenvector corresponding to a smaller (by absolute value) eigenvalue.

$d = -1$; $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

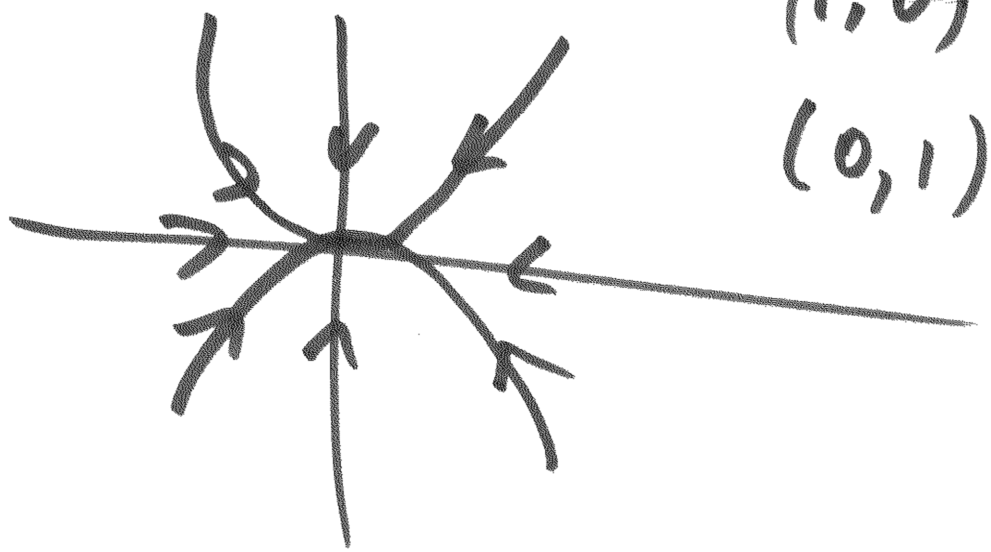
$\lambda_1 = -1 = \lambda_2$; $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x(t) = x_0 e^{-t}$

$y(t) = y_0 e^{-t} = \frac{y_0}{x_0} x(t)$



$-1 < \alpha < \emptyset$



$(1,0) \to \alpha$

$(0,1) \to -1$

$\underline{\alpha = \emptyset}$; $x(t) = x_0$

$y(t) = e^{-t}$

line of stable fixed points