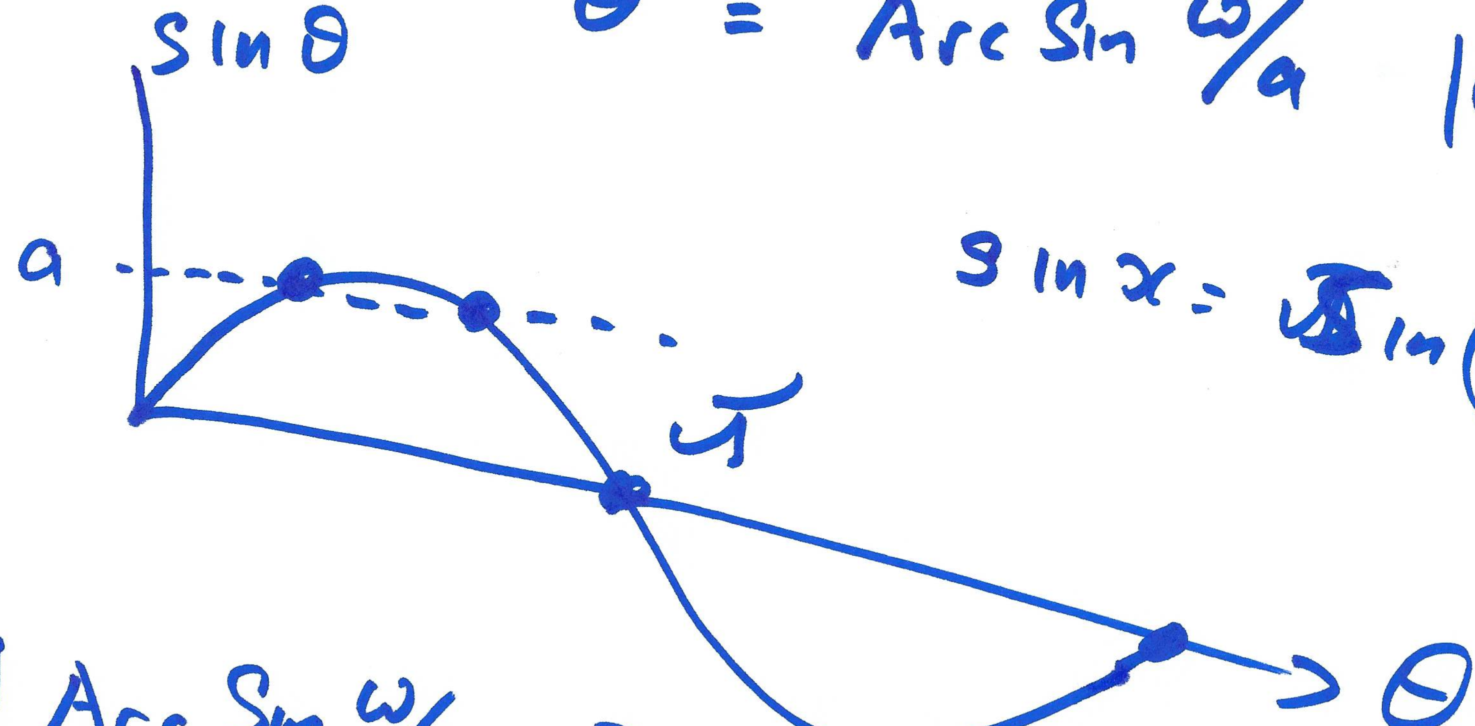
$\dot{\theta} = \omega - a \sin\theta; \qquad \omega > 0,$

Fixed points $\omega = a \sin\theta^*$

$\theta^* = \mathrm{ArcSin}\, \omega/a \qquad |a| > |\omega|$



$\sin x = \sin(\pi - x)$

$\{\mathrm{ArcSin}\, \omega/a,\ \pi - \mathrm{ArcSin}\, \omega/a\}$

$$mL\ddot{\theta}(t) + \tilde{b}\dot{\theta}(t) + mg\sin\theta = \emptyset \Big\} \times L$$

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \emptyset$$

Torque

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

torque

$$\Downarrow$$

$$\varepsilon\ddot{\theta} + \dot{\theta} + \sin\theta = \nu, \quad \varepsilon = \frac{L^3 m^2 g}{b^2}$$

$$\underline{\varepsilon \ll 1} \quad \ddot{\theta} \to \emptyset \qquad \nu = \frac{\Gamma}{mgL}$$

$$\dot{\theta} = \nu - \sin\theta$$

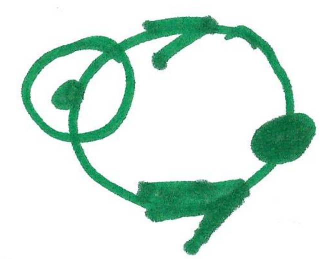
$$\dot{\theta} = \nu - \sin\theta, \quad \nu = \frac{\Gamma}{mgL}$$

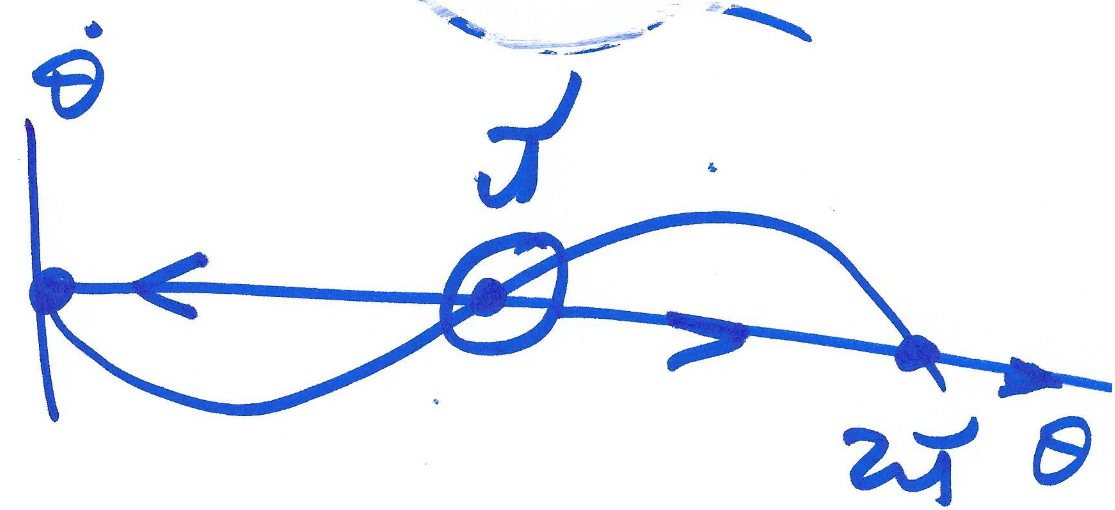
zero torque: $\nu = \emptyset$

$\dot{\theta} = -\sin\theta;$

$\theta^* = \emptyset$ - stable

$\theta^* = \pi$ - unstable



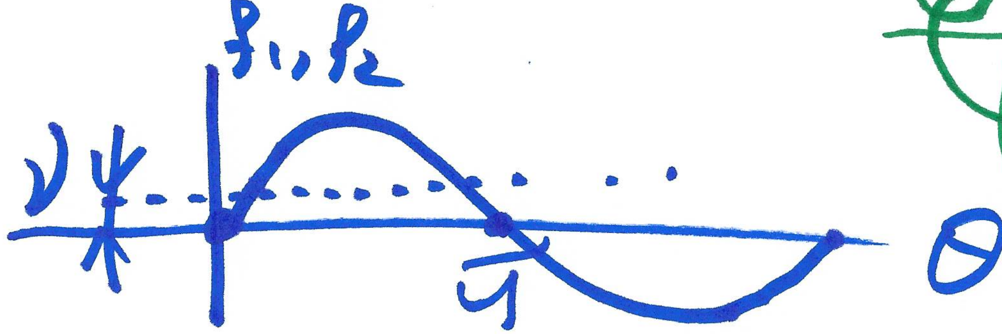


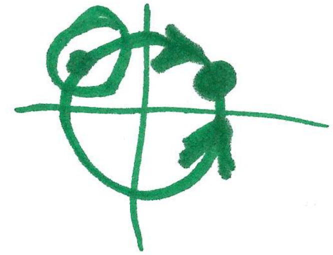
small torque : $\dot{\theta} = \nu - \sin\theta = f_1(\theta) - f_2(\theta)$

$|\nu| << 1$

$f_1(\theta) = \nu$

$f_2(\theta) = \sin\theta$





$\theta^* = \operatorname{ArcSin} \nu$

$\theta^* = \pi - \operatorname{ArcSin} \nu$

- small θ: $\sin\theta \approx \theta$

$\dot{\theta} = \nu - \theta$; $\theta^* = \nu$

<u>stable</u>;

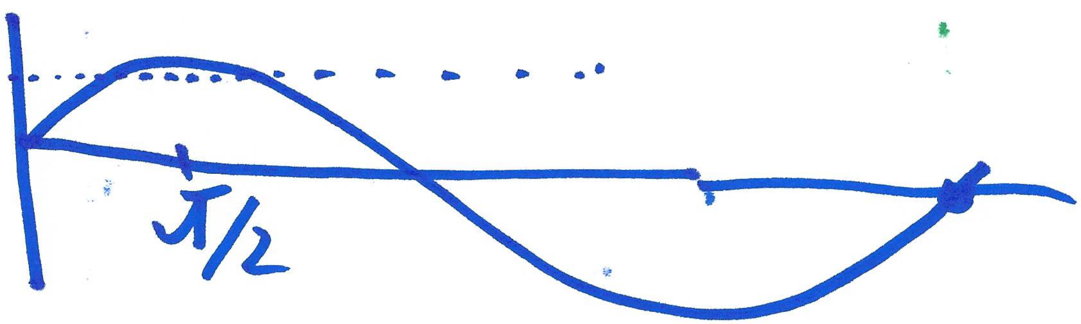
- $\theta = \pi - x$, $|x| << 1$

Equilibrium $x = \nu$ unstable

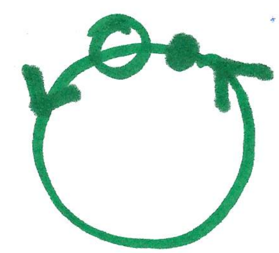
$\cancel{\theta}$ $-\dot{x} = \nu - \sin(\pi - x) = \nu - \sin x \approx \nu - x$

Larger torque:

$\nu = 1 - \varepsilon \qquad \dot{\theta} = \nu - \sin\theta$



$\nu = \frac{\Gamma}{mgL}$



$\theta = \pi/2 - x; \; \nu = 1 - \varepsilon$

$-\dot{x} = 1 - \varepsilon - \sin(\pi/2 - x) =$

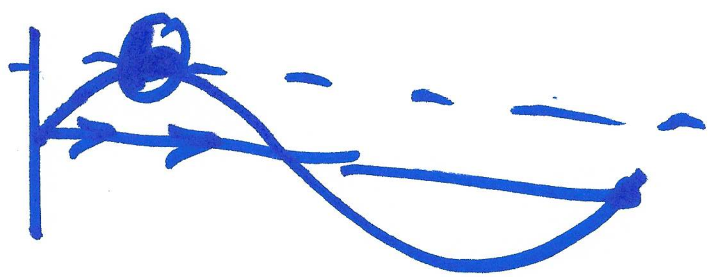
$= 1 - \varepsilon - \cos x \approx$

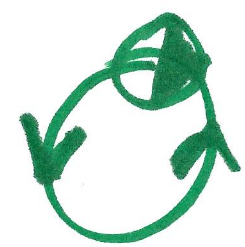
$\approx 1 - \varepsilon - (1 - x^2/2) = 1 - \varepsilon - 1 + \frac{x^2}{2}$

$\dot{x} = \varepsilon - x^2/2 \; ; \; x^* = \pm\sqrt{2\varepsilon}$

"Critical" torque $\gamma = 1$

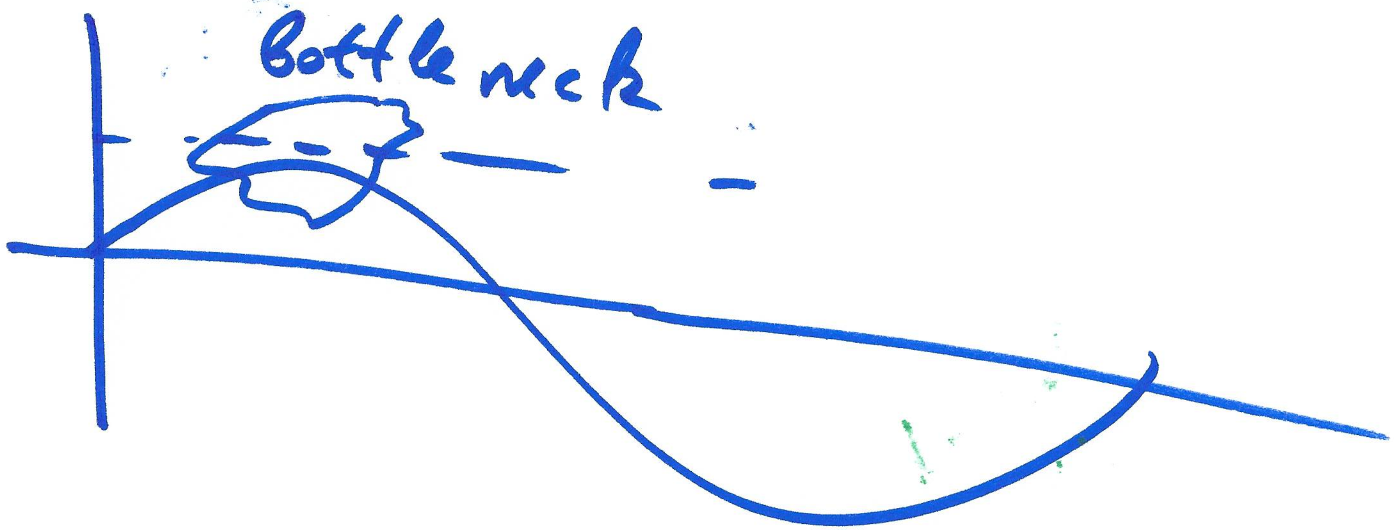
$\dot{\theta} = 1 - \sin\theta, \quad \theta^* = \pi/2;$





Bottleneck torque

$\nu = 1 + \varepsilon, \; 0 < \varepsilon << 1$

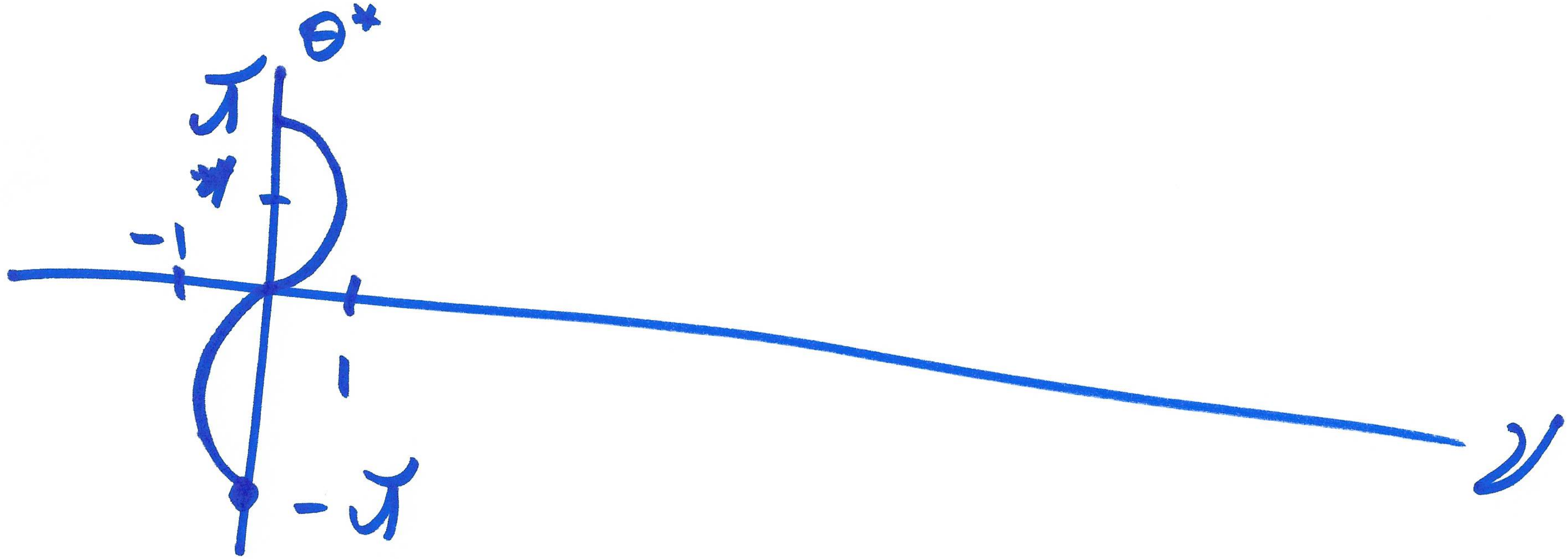


$\dot{\theta} = \nu - \sin\theta$

Large torque, $\nu \gg 1$

Bifurcation diagram

$\nu = \sin\theta^*$



Fireflies

one firefly

$\dot{\theta} = \omega$; $\theta = 0$ - light occurs

$\theta = \pi$ - no light

$\theta = \theta \bmod 2\pi$

<u>many</u>

$$\dot{\theta} = \omega + A\sin(\varphi - \theta)$$

$\varphi(t)$ - external stimulus

$\dot{\varphi} = \Omega$

ω - eigenfrequency of firefly

Ω - frequency of stimulus

A - strength of coupling

$$\dot{\varphi} = \Omega$$

$$\theta = \omega + A \sin(\varphi - \theta)$$

$$\dot{\varphi} - \dot{\theta} = \Omega - \omega - A \sin(\varphi - \theta)$$

$$\psi \equiv \varphi - \theta \qquad \dot{\psi} \equiv \frac{d}{dt}\psi(t)$$

$$\dot{\psi} = \Omega - \omega - A \sin \psi$$

$$t = T/A; \ [t] = [A] = [\text{time}^{-1}]$$

$$[T] = 1$$

$$A \frac{d}{dT}\psi(T) = \Omega - \omega - A \sin \psi$$

$$\dot{\theta}_k = \omega_k + A_k \sin(\varphi - \theta_k)$$

$$k = 1 \ldots N$$

$$\frac{d}{dT}\psi(T) = \mu - \sin\psi(T) \quad \cdot\cdot$$

$$\mu = \frac{\Omega - \omega}{A}$$

individual

$$\psi = \varphi - \theta$$

medium

Shine in unison:

fixed point: $|\mu| < 1$

$$-1 < \frac{\Omega - \omega}{A} < 1$$

$$\omega - A < \Omega < \omega + A$$

System of linear ODE

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{d}{dt}\,\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

$$\left.\begin{aligned} \dot{x}(t) &= a\,x(t) + b\,y(t) \\ \dot{y}(t) &= c\,x(t) + d\,y(t) \end{aligned}\right\}$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix}$$

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\, \underline{x}(t)$$

$$\frac{d^2}{d\tau^2} x(\tau) + x(\tau) = \emptyset$$

$$v(\tau) = \frac{d}{d\tau} x(\tau)$$

$$\frac{d}{d\tau} x(\tau) = v(\tau)$$

$$\frac{d}{d\tau} v(\tau) = -x(\tau)$$

$$\frac{d}{d\tau} \begin{pmatrix} x(\tau) \\ v(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x(\tau) \\ v(\tau) \end{pmatrix}$$

$$\underline{x}(\tau) = \begin{pmatrix} x(\tau) \\ v(\tau) \end{pmatrix}$$

$$\frac{d}{d\tau}\,\underline{x}(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{x}(\tau)$$

$$\underline{v}(\tau) = \frac{d}{d\tau}\,\underline{x}(\tau) = \begin{pmatrix} \frac{d}{d\tau} x(\tau) \\ \frac{d}{d\tau} v(\tau) \end{pmatrix} =$$

$$= \begin{pmatrix} v(\tau) \\ -x(\tau) \end{pmatrix}$$

$$\underline{x}(\tau) = \begin{pmatrix} x(\tau) \\ v(\tau) \end{pmatrix}$$

$$\frac{d}{d\tau}\,\underline{x}(\tau) = v(\tau) = \begin{pmatrix} v(\tau) \\ -x(\tau) \end{pmatrix}$$

$$\underline{x}^{T}(\tau) = (x(\tau), v(\tau)) \qquad \|\underline{x}\|_2 = \|\underline{v}\|_2$$

$$\underline{x}^{T}(\tau) \cdot v(\tau) = (x(\tau), v(\tau)) \begin{pmatrix} v(\tau) \\ -x(\tau) \end{pmatrix}$$

$$= x v - v x = \emptyset$$

$\underline{v} \perp \underline{x}$ velocity is $\perp$ to $\underline{x}$