$$\dot{\theta}(t) = \omega - a \sin \theta(t); \quad a \geq 0$$

-1

Find equilibrium points:

$$\omega = a \sin \theta^*$$

$$\theta^* = \text{Arc}\sin \omega / a$$

These points exist only if $a > \omega$

Stability:

$$\theta(t) = f(\theta);$$

$$f'(\theta) = -(\cos \theta) \cdot a$$

-2

$$\sin\theta^* = \omega/a$$

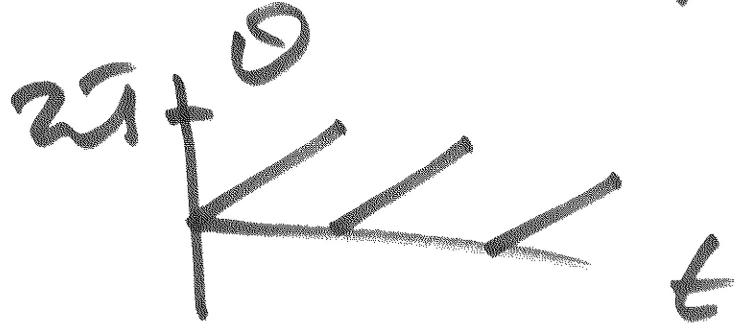
$$\cos\theta^* = \pm\sqrt{1-\sin^2\theta^*} =$$

$$= \pm\sqrt{1-(\omega/a)^2}, \quad a > \omega$$

$$f'(\theta^*) = -a\cos\theta^* = \mp\sqrt{1-(\omega/a)^2}\cdot a$$

one stable,
one unstable

$= \emptyset$; $\dot{\theta} = \omega$, $\theta(t) = \omega t + \theta_0$;

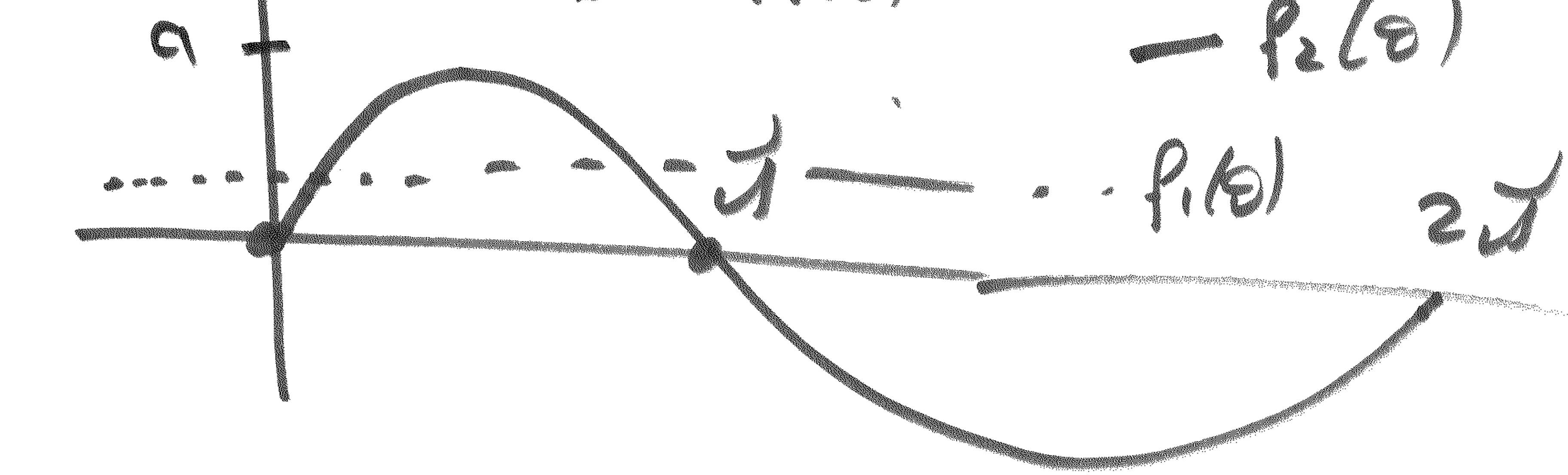


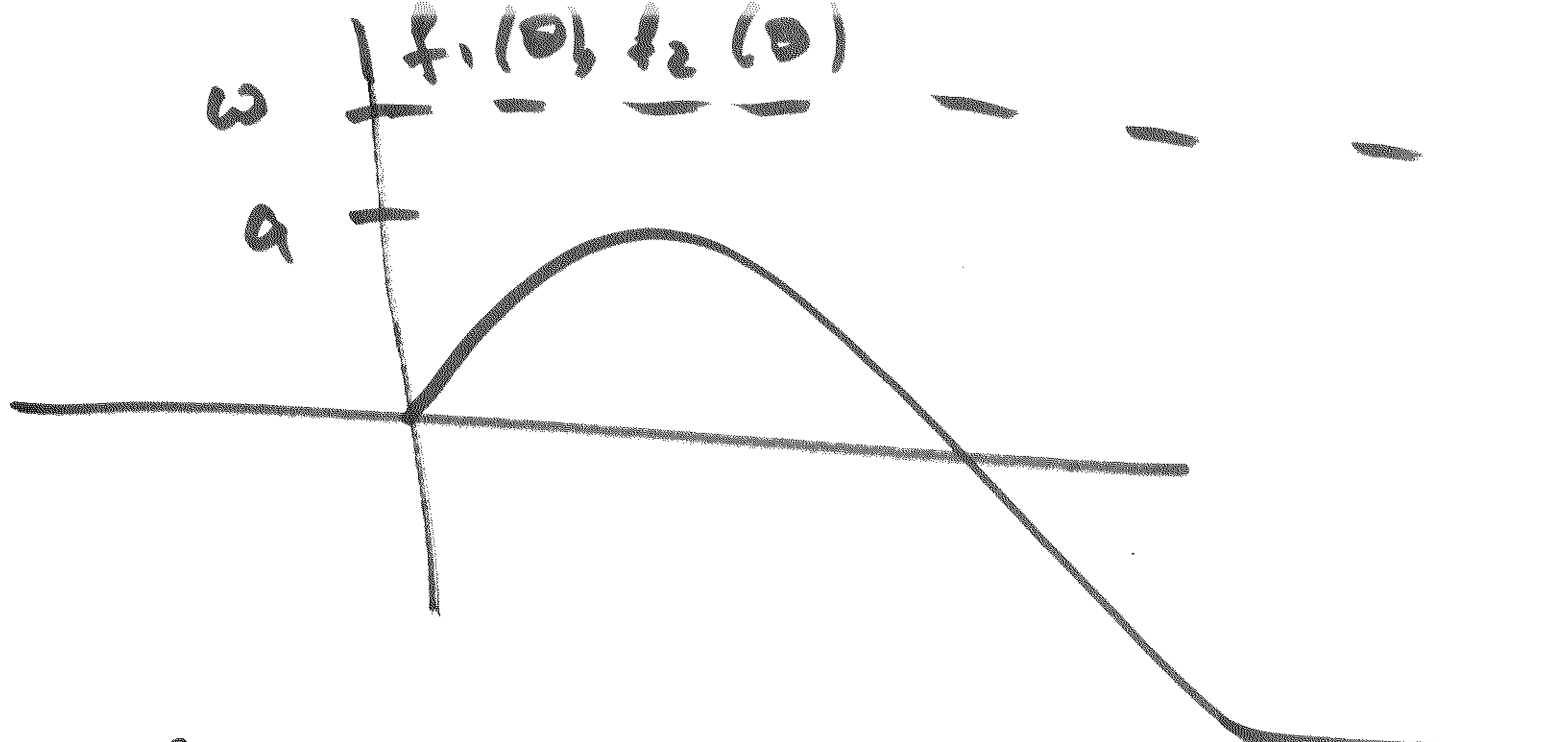
$f_1(\theta) = \omega$

$f_2(\theta) = a \sin\theta$

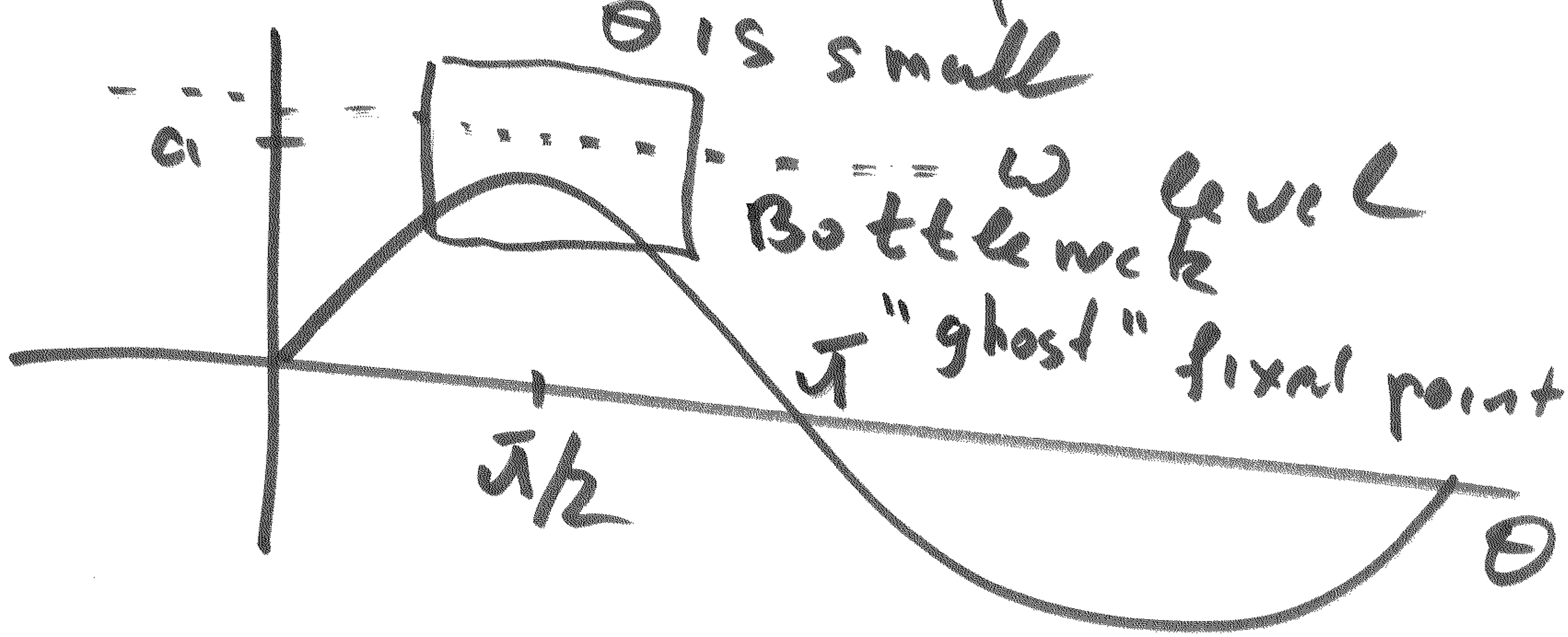
$\emptyset < a < \omega$

$\dot{\theta} = f_1(\theta) - f_2(\theta)$, no FP

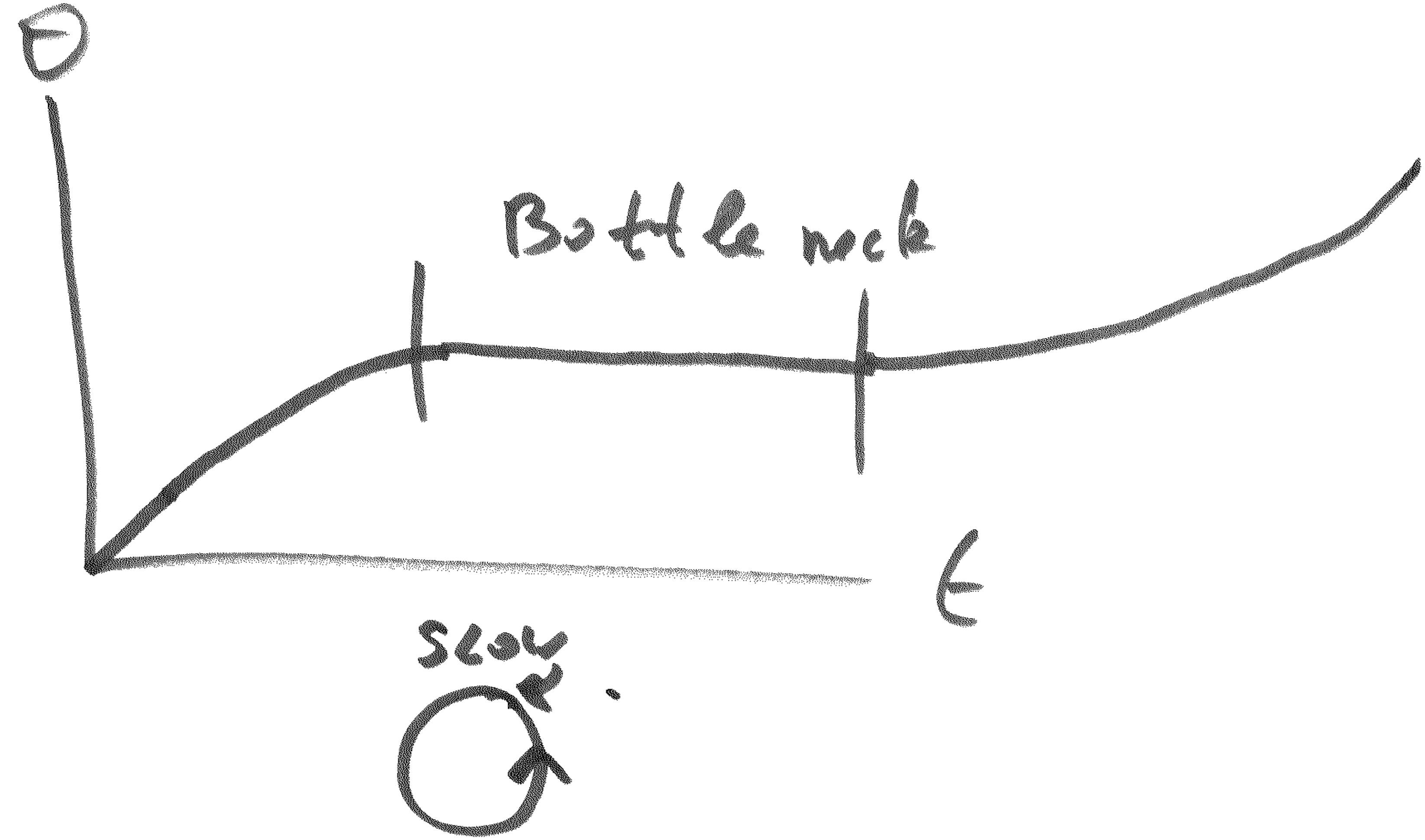




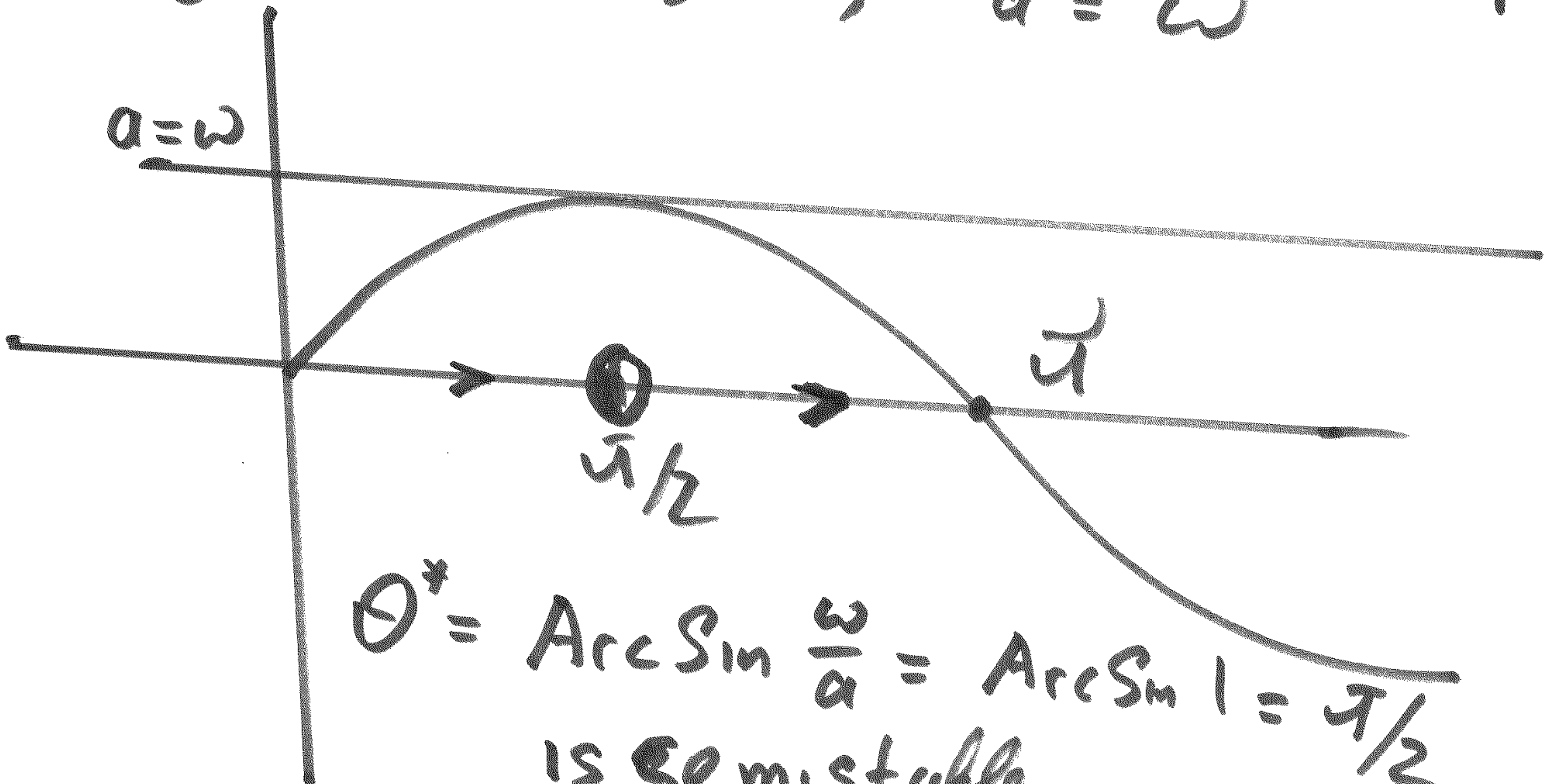
$a = \omega - \varepsilon, \quad \varepsilon << 1$



$$\dot{\theta} = \omega - a \sin\theta,$$
$$a = \omega - \varepsilon$$

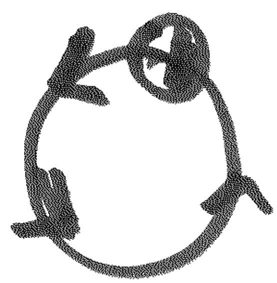


$$\dot{\theta} = \omega - a \sin \theta, \quad a = \omega$$



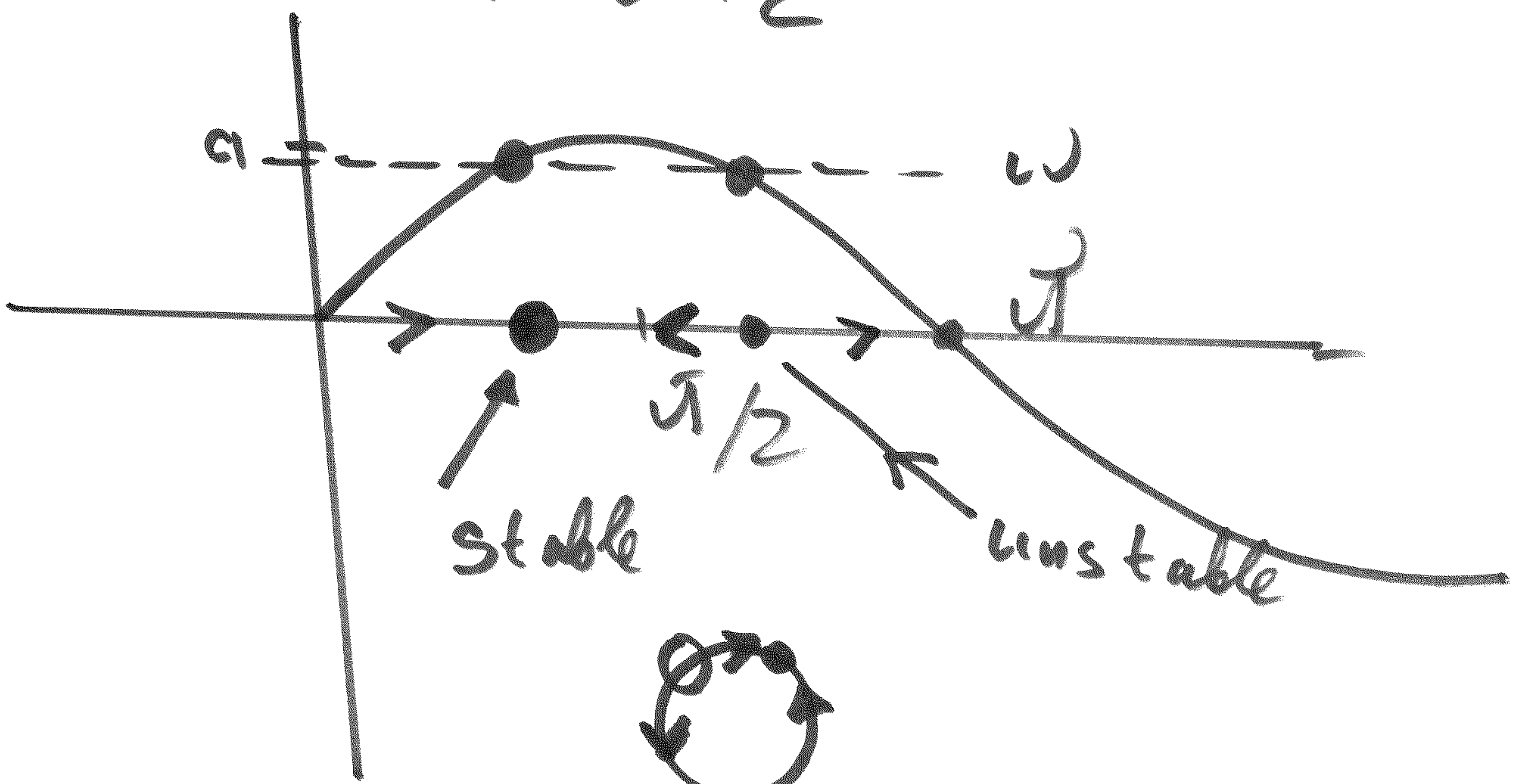
$$\theta^* = \operatorname{ArcSin} \frac{\omega}{a} = \operatorname{ArcSin} 1 = \pi/2$$

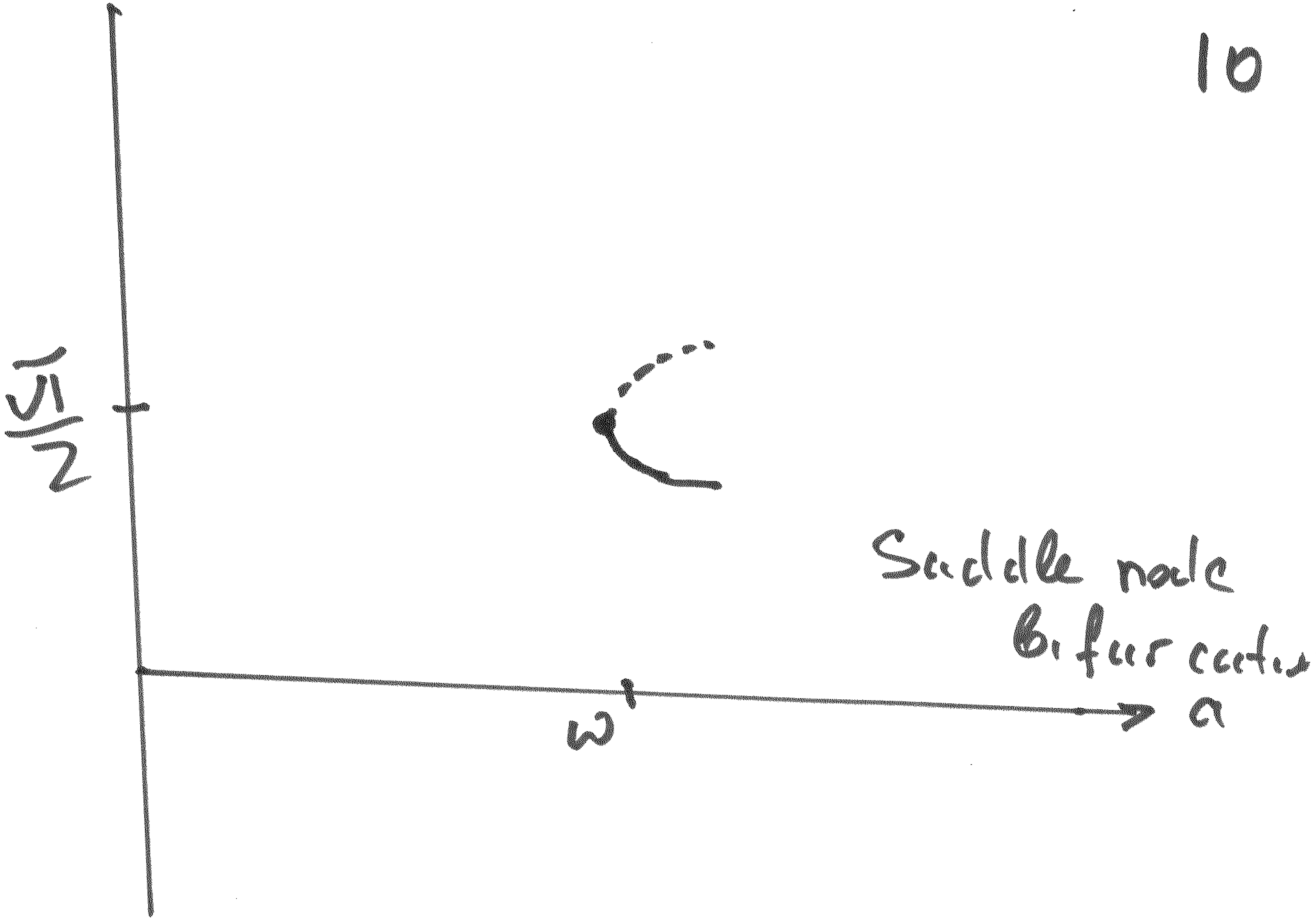
is semistable



$$\dot{\theta} = \omega - a \sin \theta,$$

$$a = \omega + \varepsilon$$




Saddle node
Bifurcation
a
3

$$\dot{\theta} = \omega - a \sin\theta, \quad 0 < a < \omega$$

Period of oscillation

$$T = \int_0^T dt = \int_0^{2\pi} d\theta \frac{dt}{d\theta}$$

$$\frac{d\theta}{dt} = \omega - a\sin\theta, \quad \frac{dt}{d\theta} = \frac{1}{\omega - a\sin\theta}$$

$$T = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

$a = 0$: $T = \dfrac{2\pi}{\omega}$

Periodic, if $|a| < \omega$

$a = \omega - \varepsilon$, $0 < \varepsilon << 1$ Square root law

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}} = \frac{2\pi}{\sqrt{\omega + a}\sqrt{\omega - a}} = \frac{2\pi}{\sqrt{2\omega}\sqrt{\varepsilon}}$$

$$\dot{\theta} = \omega - a \sin\theta$$

$$a = \omega - \varepsilon, \quad 0 < \varepsilon << 1;$$

$$\theta = \frac{\pi}{2} + x;$$

$$\dot{\theta} = \dot{x}$$

$$\dot{x} = \omega - (\omega - \varepsilon)\sin\left(\frac{\pi}{2} + x\right)$$

$$\dot{x} = \omega - (\omega - \varepsilon)\cos(x)$$

$$\dot{x} = \omega - (\omega - \varepsilon)\left(1 - \frac{x^2}{2}\right); \quad , \; |x| << 1$$

$$\cos x = 1 - x^2/2$$

$$\dot{x} = \omega - (\omega - \varepsilon)\left(1 - \frac{x^2}{2}\right)$$

$$= \omega - \left(\omega - \frac{\omega x^2}{2} - \varepsilon + \varepsilon \frac{x^2}{2}\right)$$

$$= \omega - \omega + \frac{\omega x^2}{2} + \varepsilon + \cancel{\frac{\varepsilon x^2}{2}}$$

$$\dot{x} = \varepsilon + \frac{\omega x^2}{2}$$

$0 < \varepsilon \ll 1$

$|x| \ll 1$

$$x = \frac{2y}{\omega}$$

$$\frac{2\dot{y}}{\omega} = \varepsilon + \frac{\omega}{2} \cdot \frac{4 \cdot y^2}{\omega^2}$$

$$\dot{y} = \frac{\varepsilon \omega}{2} + y^2$$

Normal form for saddle node

$$\dot{y} = \frac{\varepsilon\omega}{2} + y^2; \quad r \equiv \frac{\varepsilon\omega}{2}$$

$$\dot{y} = r + y^2;$$

$$T = \int dt$$

$$y =$$

Overdamped pendulum with a torque

$$mL\ddot{\theta} + \nu\dot{\theta} + mg\sin\theta = 0$$

$$\Downarrow$$

$$mL^2\ddot{\theta} + \nu L\dot{\theta} + mgL\sin\theta = 0$$

Add Torque Γ, $\nu L = b$

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

$t = T \cdot \gamma;$

$[t] = [T]; \quad [\gamma] = 1$

$$\frac{mL^2}{T^2}\frac{d^2}{d\tau^2}\theta(\tau) + \frac{b}{T}\frac{d}{d\tau}\theta(\tau) + mgL\sin\theta = \Gamma$$

$$\frac{\frac{L}{T^2}}{g}\frac{d^2}{d\tau^2}\theta(\tau) + \frac{b}{mgLT}\frac{d}{d\tau}\theta(\tau) + \sin\theta = \frac{\Gamma}{mgL}$$

Choose T so that

$$b = mgLT,$$

$$T = \frac{b}{mgL}$$

$$\frac{L^3 m^2 g}{b^2} \frac{d^2}{d\tau^2}\theta(\tau) + \frac{d}{d\tau}\theta(\tau) + \sin\theta = \frac{\Gamma}{mgL}$$

$$\varepsilon = \frac{L^3 m^2 g}{b^2} \; ; \; \nu = \frac{\Gamma}{mgL}$$

$$\varepsilon \frac{d^2}{d\tau^2}\theta + \frac{d}{d\tau}\theta(\tau) + \sin\theta = \nu$$

$\varepsilon \ll 1$,

$$\frac{d}{d\tau}\theta(\tau) = \nu - \sin\theta(\tau)$$

$$[b] = [mgL \cdot T] = M \cdot \frac{L^2}{T^2} \cdot T = ML^2/T$$

$$\varepsilon = \frac{L^3 m^2 g}{\sigma^2}$$

$$[\varepsilon] = \left[\frac{L^3 M^2 L}{T^2} \frac{T^2}{M^2 L^4} \right. = 1$$