Ordinary Differential Equations and Dynamical Systems
MATH 4400 FALL 2024

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Tuesday-Friday 12:00am
Office Hours : Monday 18-20, Webex meeting room lvovy
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August 29, 2024

Midterms are on October 1, November 1 and December 3, 2024
T
here will be no lecture on December X6 the make up lecture will be on November 20
If you are late to class, you agree to either dance, tell a joke or sing.

An intermediate course emphasizing a modern geometric approach and applications in science and engineering. Topics include first-order equations, linear systems, phase plane, linearization and stability, calculus of variations, Lagrangian and Hamiltonian mechanics, oscillations, basic bifurcation theory, chaotic dynamics, and existence and uniqueness.

1 Learning Outcomes:

1. The students will be able to solve problems in ordinary differential equations and dynamical systems.
2. The students will be able to write coherent mathematical and scientific arguments needed in solving ordinary differential equations and related application problems in science and engineering.
3. The students will be able to use tools from differential equations to provide approximate or exact solutions for problems arising in physics, scientific and engineering applications.
4. The students will be able to choose the appropriate techniques from variational calculus to find e exact and qualitative solutions of differential equations.

2 Outline

1. Chapter ONE, INTRODUCTION: What will you get out of the course
2. Chapter TWO, FLOWS on the line: Solving

$$\dot{x}(t) = f(x(t)).$$

3. Chapter THREE, BIFURCATION: elementary biffurcation theory
4. Chapter FOUR, FLOWS on a circle
5. Chapter FIVE: LINEAR SYSTEMS
6. Chapter SIX: PHASE PLANE
7. Chapter SEVEN: LIMIT CYCLES
8. The End?
9. Chapter EIGHT: Biffurcations Revisited
10. Chapter NINE:Lorenz equations
11. Chapter TEN: Maps, fractals, Strange attractors

This is approximate, will modify as we go along.

BOOK S. H. Strogatz, Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering, Addison-Wesley.

3 Grade Policy

Homework projects **30 %** + two best Midterm Exams **35 %** + Final Exam+ **35**

Optional, subject to availability: Final Project +**10 %**

"Cell Phone" usage in class -5 percent of the final grade

I will assign a grade based on the following:

$90 \leq g \leq 100$ is A
$80 \leq g < 90$ is B
$70 \leq g < 80$ is C
$67 \leq g < 70$ is D
$g < 67$ is F

Grades modifiers will be used: $0 \leq g \leq 3$ is "-", $3 < g \leq 6$ has no modifier, $6 < g$ is "+".

There is no "make up" policy for homeworks.

Midterm make up policy If you miss one midterm, you will get grade zero for that midterm, and that would be the grade that is removed from your score as described in the grade policy. The make up policy for the second missed midterm is a Russian-style oral exam on the material covered. You are randomly given one paragraph title to present, and, after given time to prepare

(20 minutes or so), you present the material of that paragraph to the professor. You are asked questions to probe the level of your understanding, and then you are given a problem to solve.

ALL HOMEWORKS ARE REQUIRED

Late Homeworks: 10 % off per each full 24 hours. *This policy is strictly enforced*

I did hear stories about dogs eating homeworks.

Please submit HW via LMS

There is no "make up" policy for homeworks or exams.

Time line for final project:

- **stage one** - you think about what you want to do, talk to your friends, talk to me, bounce ideas of the wall - September 15, 2023
- **stage 2**- you submit to me a title and a one page description of your project September 30, 2023
- **stage 3** I choose 3 or 4 best projects and will notify you if you are selected.
- **stage 4** - you give me your talk November 15, 2022
- **stage 4** you give in-class presentation and answer questions December 1, 2023
- Last day of classes - you give me your written report in LATEX format

To be eligible for final project your grade, with out the final project, should be A,B or C.

3.1 Few notes, policies, etc

There is no "curve grading".

Attendance is strongly encouraged and may be **logged** You are responsible for knowing what was said in class. Use class webpage at your own risk.

Usage of **LAPTOPS, CELL PHONES, ipods or lightsabers.** will result in 5 percent substracted from your final grade.

Please Submit your HW on LMS.

My promise: Those with passing grade will be proficient with Dynamical Systems.

Academic Integrity

The grade you receive for the course will be based on the work that you do.

With this principle in mind, the work (exams, homework, computer programs) that you present for a grade MUST, in fact, be of your own.

With respect to the exams, this means that no assistance or collaboration of any kind is permitted (other then assisace obtained from the instructor). Anyone violating this policy will receive an exam grade of zero and will be reported to the Dean of Students.

With respect to problem sets (homeworks and or computer projects), you are free to seek assistance or advise from any person, book or computer. However, what you hand in must be your own work. In this regard, **computer files must not be shared or exchanged** nor should you copy work from someone else. If you do collaborate, please **provide names** of people you worked with at the beginning of your assignment. Violating of this policy will result in a score of zero for the assignment or the exam, and will be reported to the Dean of Students.

Second violation will get you grade F for the course.

Note that your health, need of financial aid, need to maintain GPA, need to graduate, obtain employment, etc **will not** be considered.

BUT if you sense you have a problem, please talk to me. I may be able to help!

3.2 Important Note on HW's

It is your responsibility to present your Homework in a best possible way. Please write neatly (better type), and respect grader/instructor time. Do not expect grader/instructor to figure out what you meant. Make it crystal clear!

Welcome to A

Math 4400

yuri@vov.com/ODE

B

PDE

$$\frac{\partial}{\partial t}\bar{u}(\bar{x},t) + (\bar{u}(x,t)\cdot\bar{\nabla})\bar{u}(x,t) = -\nabla p(\bar{x},t)/\rho$$

<u>ODE</u> harmonic oscillator

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = \emptyset$$

m, b, k are given constants

C

$$\frac{d}{dt} x(t) \equiv \dot{x}(t);$$

$$\begin{cases} \dot{x}_1 = f_1(x_1\ x_2\ x_3 \ldots x_n\ t) \\ \dot{x}_2 = f_2(x_1\ x_2\ x_3 \quad x_n\ t) \\ \dot{x}_n = f_n(x_1\ x_2\ x_3 \quad x_n\ t) \end{cases}$$

"coma's are implied"

System of first order ordinary differential equations

D

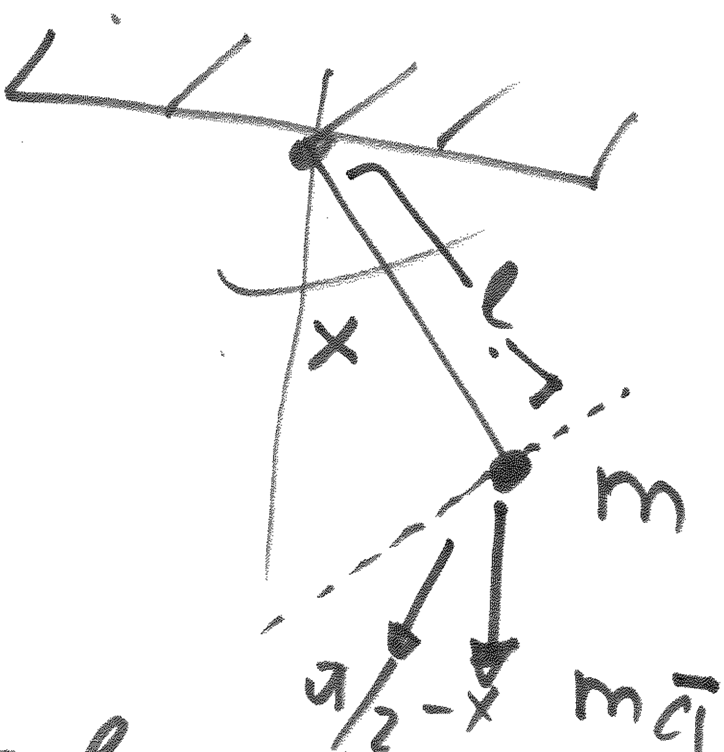
$$m\ddot{x} + b\dot{x} + kx = \emptyset;$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \emptyset;$$

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -\frac{b}{m}y(t) - \frac{k}{m}x(t) \end{cases}$$

n'th order ODE is equivalent to a system of n' ode's of first order.

$$m l \ddot{x}(t) + mg \sin(x(t)) = 0$$ ⇐



l - length - meters
m - mass - kg
g - gravity acceleration

$$[g] = \frac{\text{meters}}{\text{sec}^2}$$

$$\ddot{x}(t) + \frac{g}{l}\sin x(t) = 0$$ F

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -\frac{g}{l}\sin x(t) \end{cases}$$

$$|x(t)| \ll 1$$

$$\sin x \simeq x - ~~\frac{x^3}{6}~~ ~~\frac{x^5}{120}~~$$

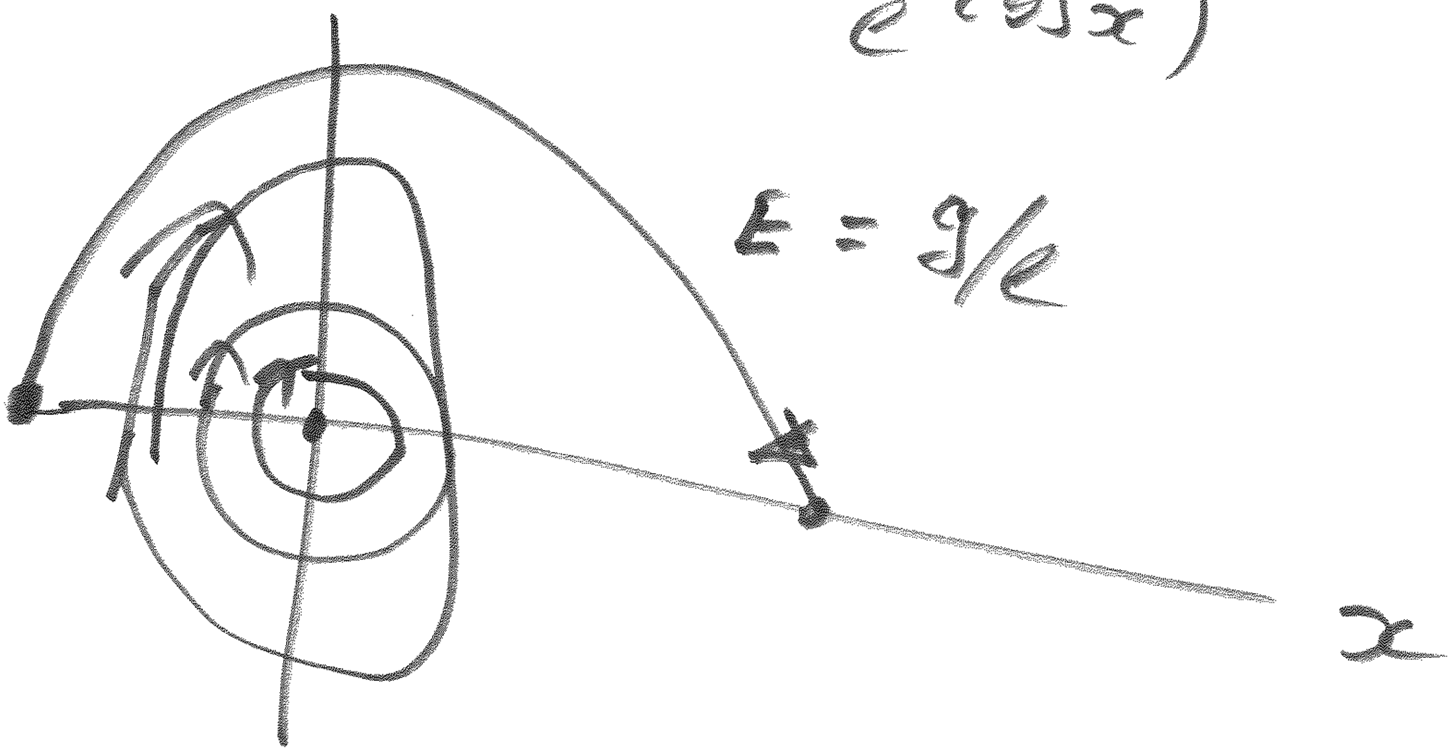
$$\ddot{x} + \frac{g}{l}x = 0 \qquad g/l = \omega^2$$

Solution is not valid for large value of x.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g}}\sqrt{l}$$

$$\dot{x}^2 = 2\left(E + \frac{g}{\ell}\cos x\right)$$

$$\dot{x} = \pm\sqrt{2\left(E + \frac{g}{\ell}\cos x\right)}$$



$\tan x/2 = e^{t+c}$

$x = 2\,\mathrm{ArcTan}\left(e^{t+c}\right)$

+ explicit analytical solution

- what happens at $t \to \infty$
- let $x(t) = \pi/2$; what happens?

Phase Plane technique.

F

$$\frac{dx}{dt} = \sqrt{2\left(E + \frac{g}{e}\cos x\right)}$$

$$\int dt = \int \frac{dx}{\sqrt{2\left(E + \frac{g}{e}\cos x\right)}}$$

$E > \frac{g}{e}$ - solution exists for all values of x

$E < g/e$ → solution is confined for small values of x

$E - \frac{g}{e}\cos x = 0$,

$x = \text{Arc Cos}\,\frac{Ee}{g}$

$\bar{E}$

$$\ddot{x}(t) = -\frac{g}{\ell}\sin(x(t))\,; \quad \Big]\ \dot{x}(t)$$

$$\ddot{x}(t)\,\dot{x}(t) + \frac{g}{\ell}\,\dot{x}(t)\sin(x(t)) = \emptyset\,;$$

$$\ddot{x}(t)\,\dot{x}(t) = \frac{d}{dt}\left(\dot{x}(t)\right)^2/2$$

$$\dot{x}(t)\sin(x(t)) = -\frac{d}{dt}\cos(x(t))$$

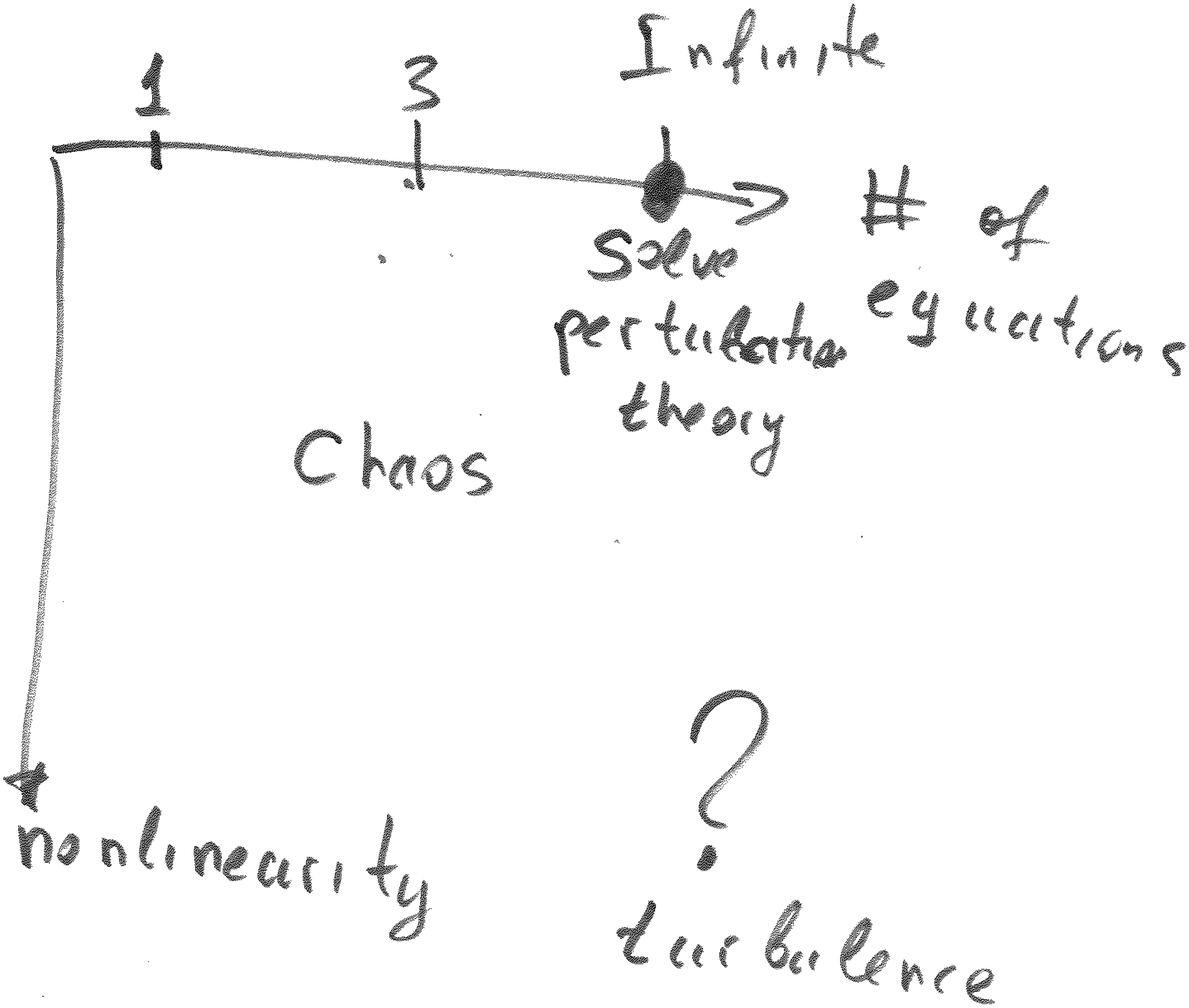
$$\frac{d}{dt}\frac{\left(\dot{x}(t)\right)^2}{2} - \frac{g}{\ell}\frac{d}{dt}\cos(x(t)) = \emptyset$$

$$\frac{d}{dt}\left[\frac{\dot{x}^2(t)}{2} - \frac{g}{\ell}\cos x(t)\right] = \emptyset$$

$$\frac{\dot{x}^2}{2} - \frac{g}{\ell}\cos x = E$$

$$\dot{x}^2 = 2\left(E + \frac{g}{\ell}\cos x\right)$$

E



1
3
Infinite
# of equations
Solve perturbation theory
Chaos
nonlinearity
?
turbulence

Chapter 2 F

Flow on a line

$\dot{x}(t) = f(x(t))$; autonomous

one equation,

ordinary differential

"system" => dynamical system

Most general form

$\dot{x}(t) = f(x(t), t)$

G

Solve

$$\dot{x}(t) = \sin(x(t))$$

Separable: $$\int \frac{dx(t)}{\sin(x(t))} = \int dt$$

$$\int \frac{dx}{\sin x} = \ln(\tan(x/2)) + C = t + C;$$

$$(\ln(\tan(x/2)))' = \frac{1}{2\tan(x/2)\cos^2(x/2)}$$

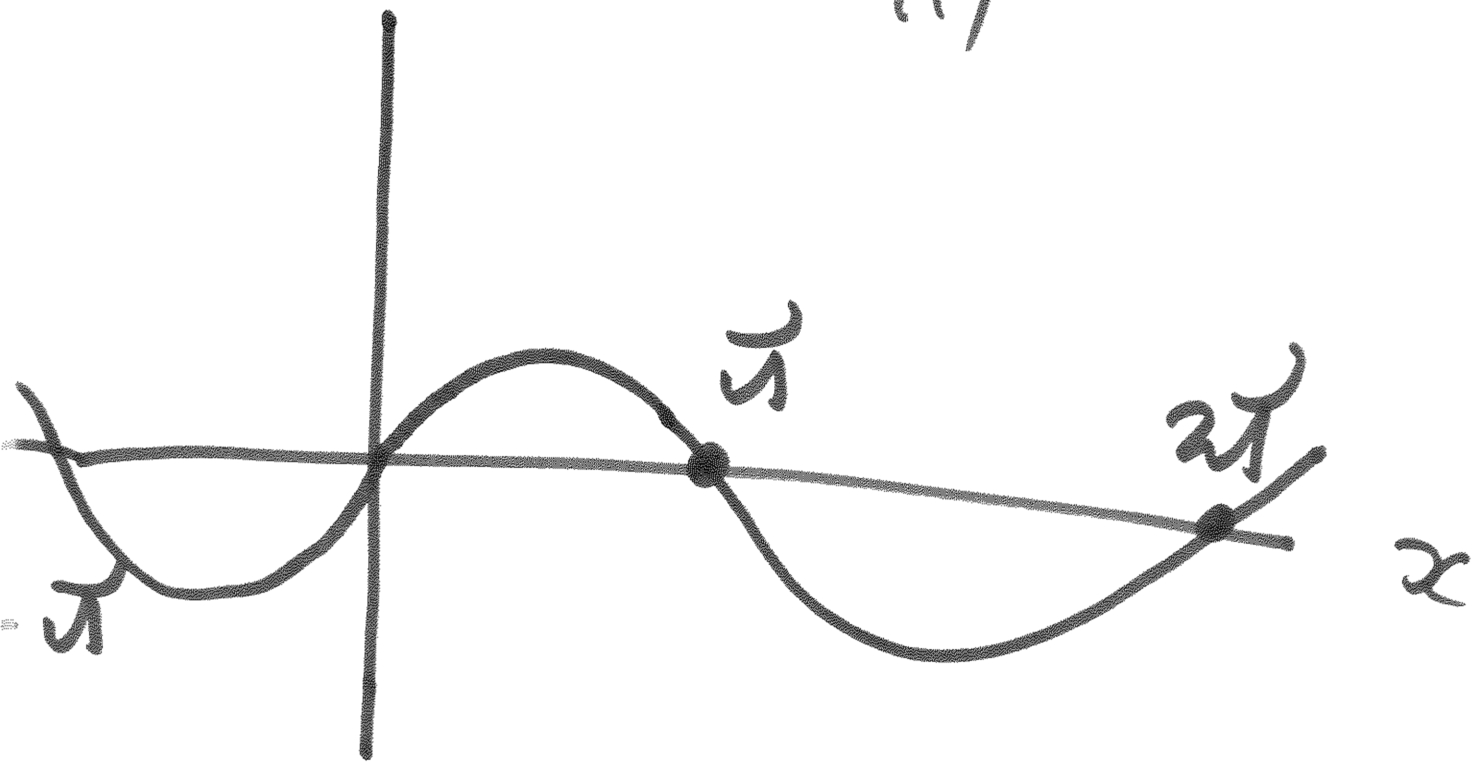
$$(\tan(x))' = \frac{1}{\cos^2 x}$$

$$= \frac{1}{2\sin x/2 \cos x/2}$$

$$= \frac{1}{\sin x}$$

I

$$\dot{x}(t) = \sin x(t)$$



Fixed points $\dot{x}(t) = 0$

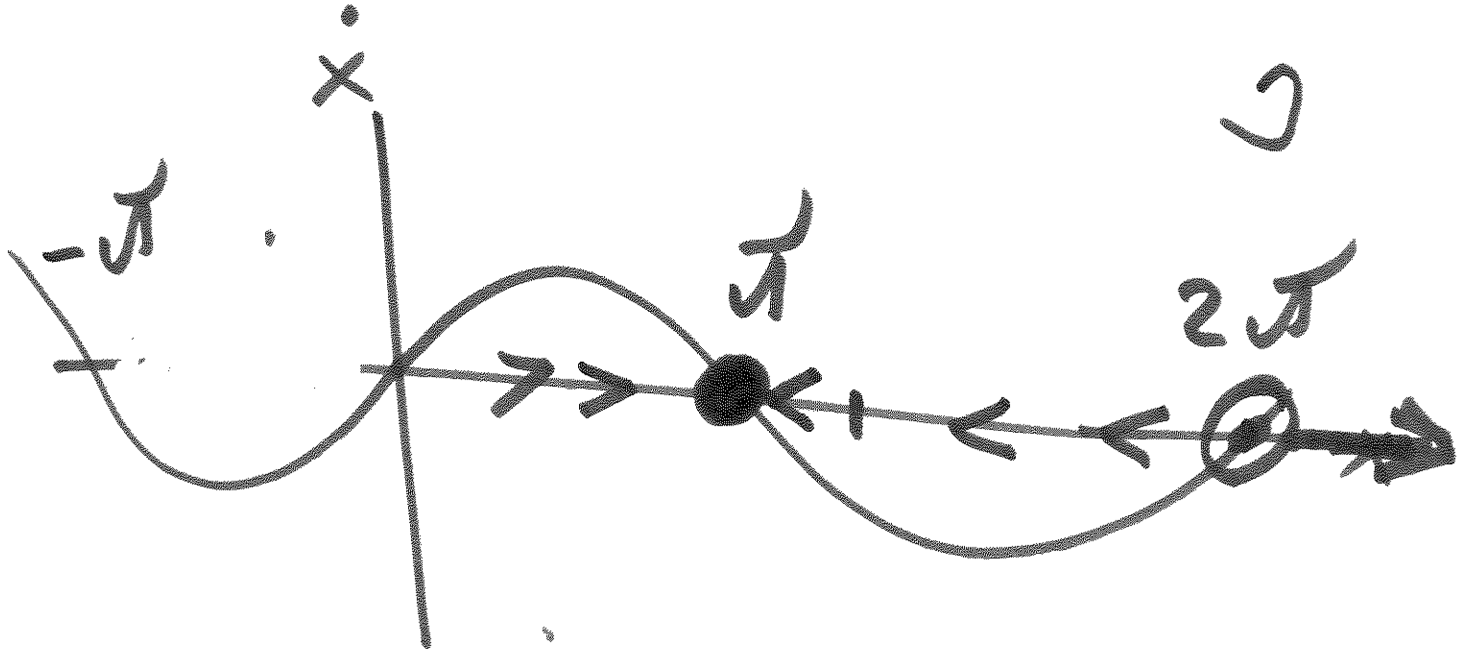
$$\dot{x} = 0 \iff x = \pi \cdot n$$

$$n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

IF $x(t=0) = n\pi$

then $x(t) = n\pi \quad \forall t$

$\forall \equiv$ for all values of



$x(t=0) = \pi + \pi/10 i$

$\dot{x}(t=0) < 0$

$x(t=0) = \theta, \; \pi < \theta < 2\pi$

$\dot{x}(t=0) < 0$

go to the left

$x = \pi$ IS STABLE

FIXED POINT

K

2π is unstable

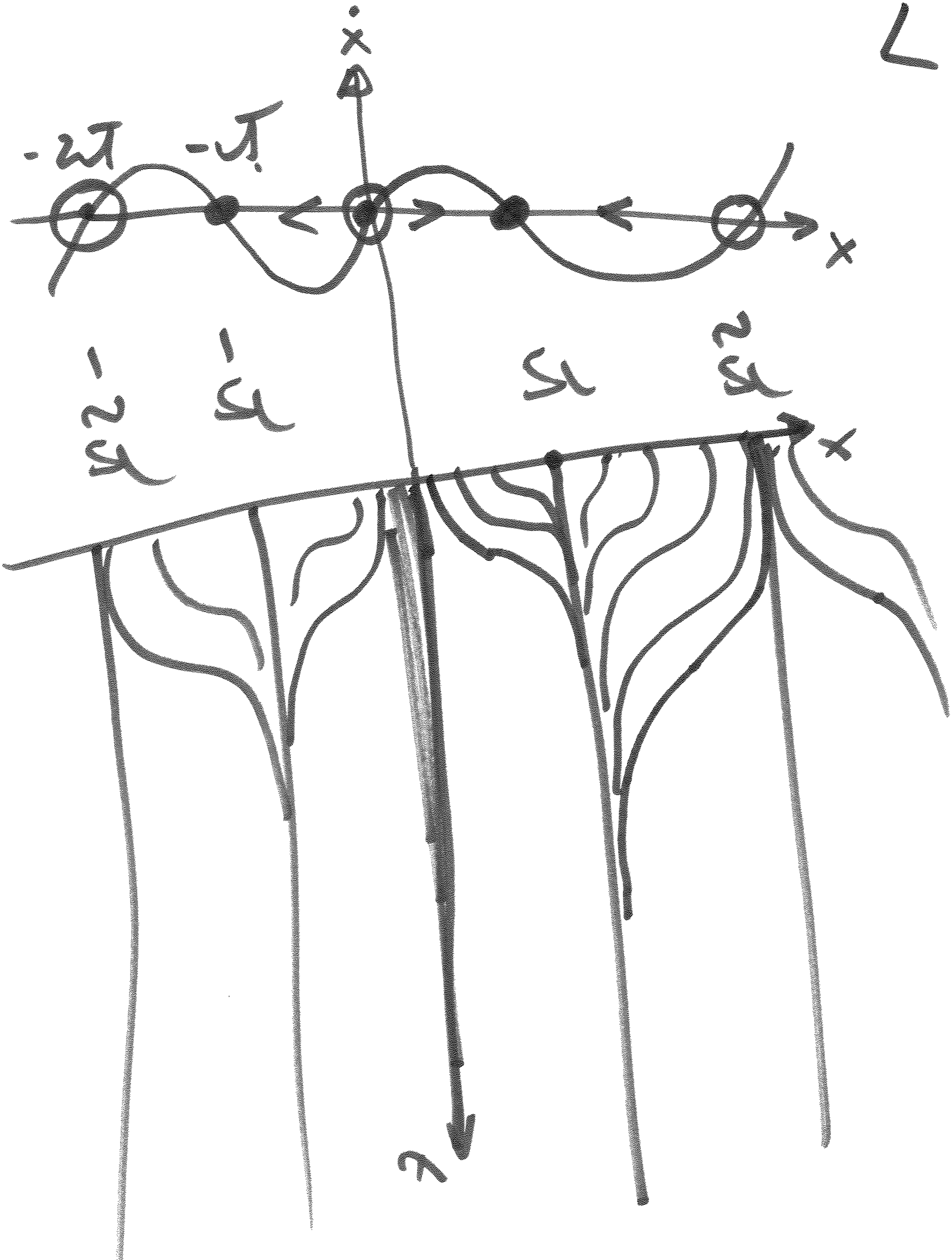
$x = 2\pi n,$

$n = 0, \pm 1, \pm 2, \pm 3 \ldots$

is unstable

fixed

point.


ẋ
-2π
-π
x
x
r

M

Algorithm:

Consider $\dot{x}(t) = f(x(t))$;

1. Find x^* such that $f(x^*) = \emptyset$
2. Plot $\dot{x}(t)$ versus x
3. IF $x(t=\emptyset) = x^*$, then $x(t) = x^* \; \forall t$
4. Consider spaces between x^*
   - IF $f(x) > \emptyset$, flow to the right
   - IF $f(x) < \emptyset$ -//- left

Classify stable and unstable fixed points: 2

- IF $f(x^*) = 0$, $f(x^* + \varepsilon) > 0$
  $f(x^* - \epsilon) < 0$

$0 < \varepsilon << 1$

then x^* is unstable

- IF $f(x^* + \varepsilon) < 0$, $f(x^* - \epsilon) > 0$

  — ıı — stable

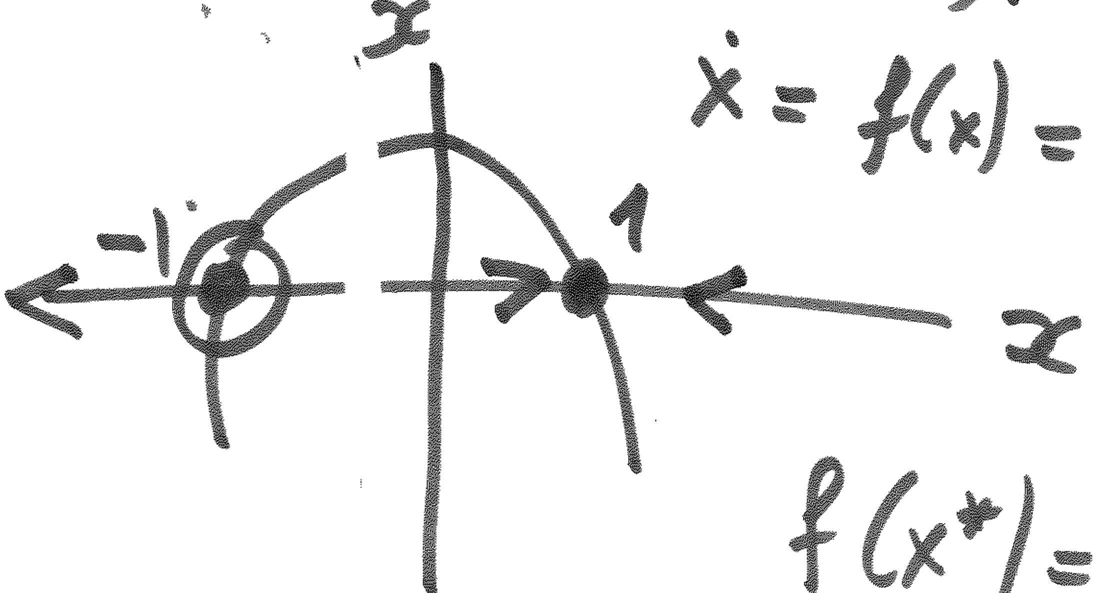
- Rotate 90°

  plot $x(t)$

Example

$\dot{x} = 1 - x^2$;

Find Fixed points, classify their stability, plot $x(t)$.



$\dot{x} = f(x) = 1 - x^2$

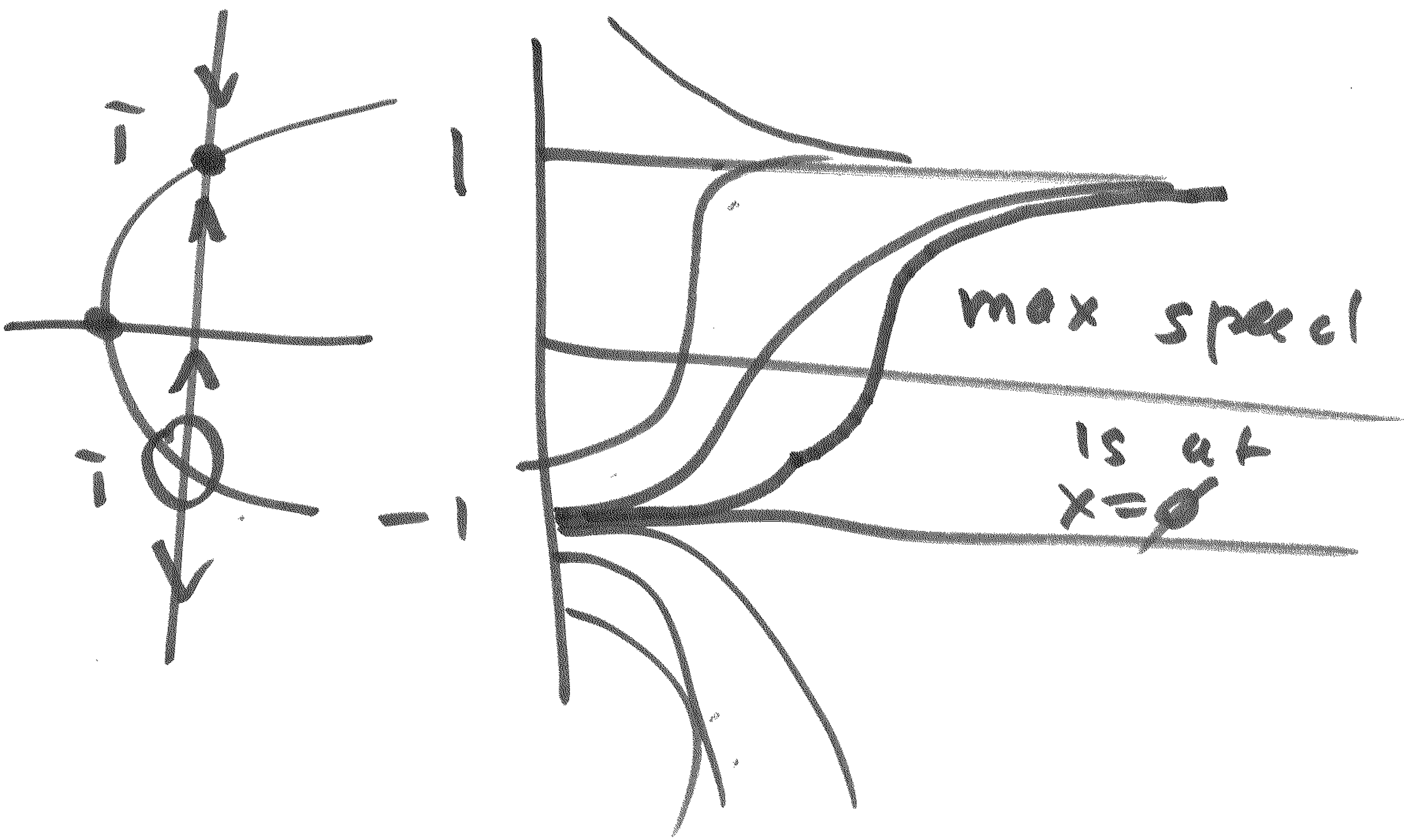
$f(x^*) = \emptyset \Rightarrow$

$x = \pm 1$

$x = 1$ is stable Fixed Point

$x = -1$ is unstable -''- ''-

e



$\dot{x} = 1 - x^2$ can be solved analytically

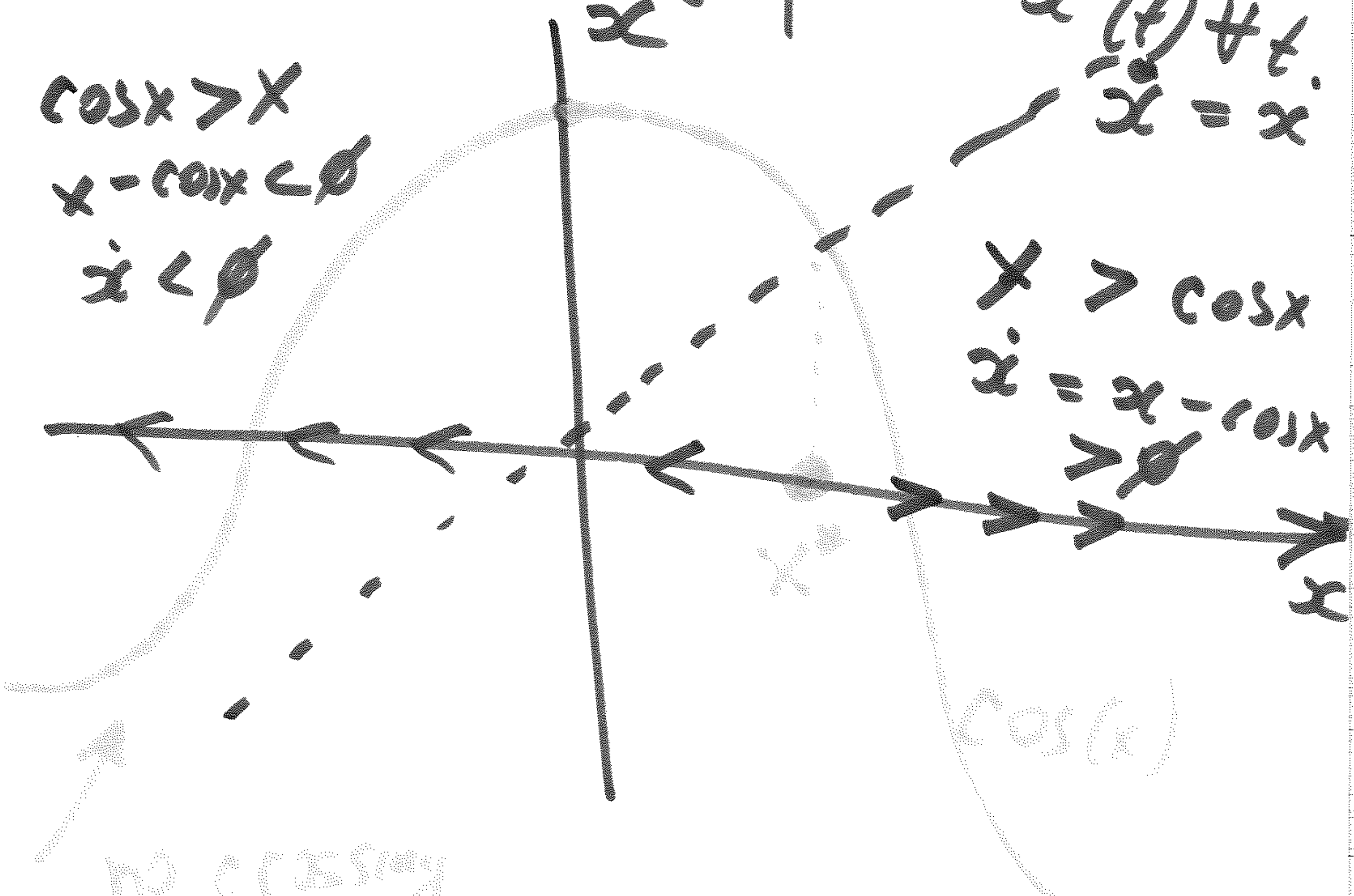
9

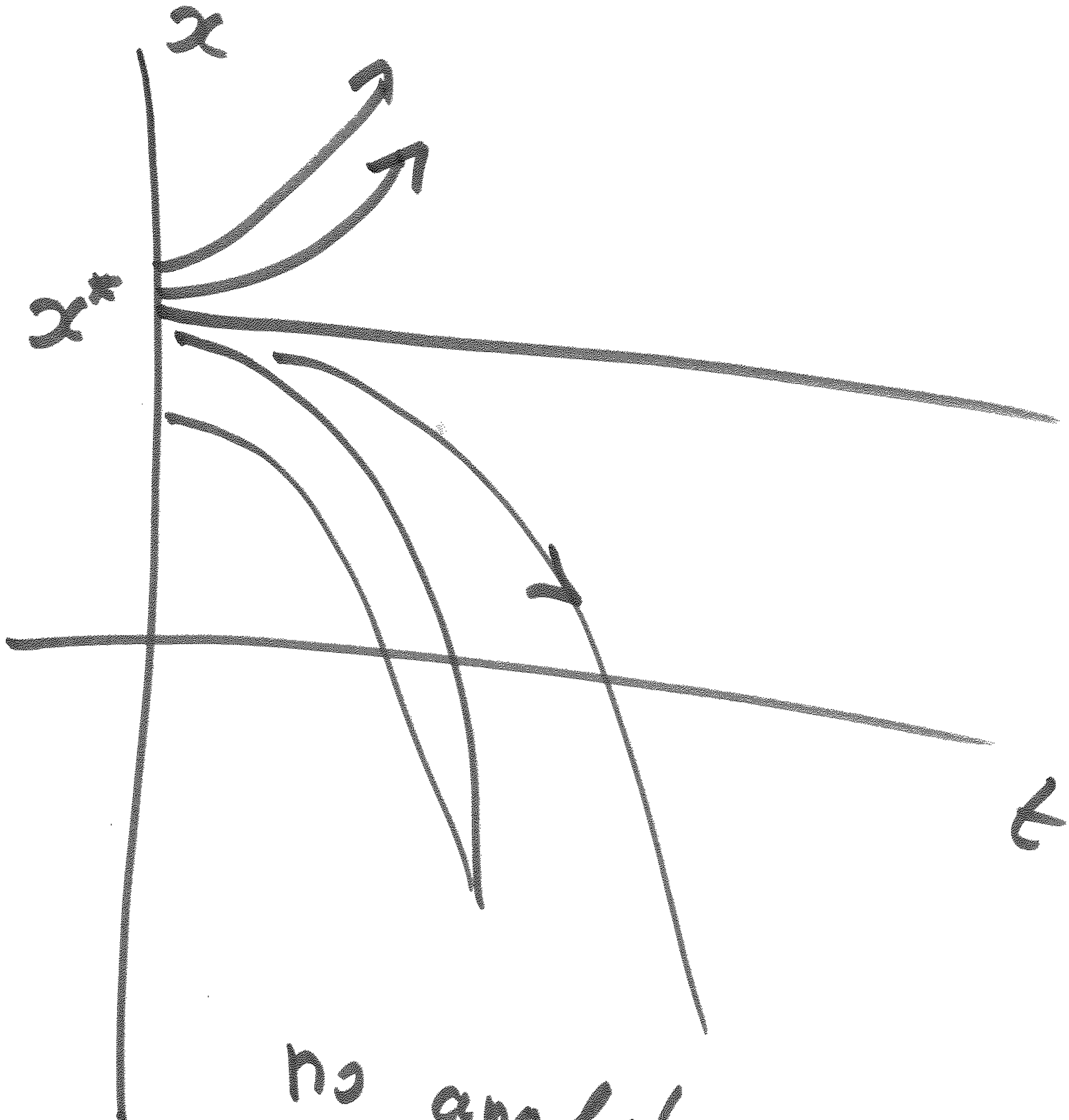
R

$\dot{x} = x - \cos x;$

$x \equiv x(t);$

Find fixed points, classify their stability, plot $x(t)\ \forall t$.





no analytical solution

$\dot{x} = f(x(t));$

x^* is fixed point

$f(x^*) = \emptyset;$

$x(t) = x^* + \eta(t);$

$\dot{\eta}(t) = f(x^* + \eta(t))$

$\approx f(x^*) + f'(x^*)\,\eta(t)$

~~$+ f''(x^*)\,\eta^2(t)/2 + \ldots\ldots$~~

$f(x^*) = \emptyset;$

$\dot{\eta}(t) = f'(x^*)\,\eta(t)$