Ordinary Differential Equations and Dynamical Systems
MATH 4400 FALL 2026

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Monday-Thursday noon **Green 120**

Office Hours : Monday 18-20, Webex meeting room lvovy

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TA is Chanaka mapamc@rpi.edu Tuesdays 16-18, AE432

August 30, 2026

Midterms are on October 8, November 2 and December 3, 2026

There will be no lecture on November 30 the make up lecture will be on November 11(?)

If you are late to class, you agree to either dance, tell a joke or sing.

Lecture Notes

PDF copy of this webpage

An intermediate course emphasizing a modern geometric approach and applications in science and engineering. Topics include first-order equations, linear systems, phase plane, linearization and stability, calculus of variations, Lagrangian and Hamiltonian mechanics, oscillations, basic bifurcation theory, chaotic dynamics, and existence and uniqueness.

1 Learning Outcomes:

1. The students will be able to solve problems in ordinary differential equations and dynamical systems.
2. The students will be able to write coherent mathematical and scientific arguments needed in solving ordinary differential equations and related application problems in science and engineering.
3. The students will be able to use tools from differential equations to provide approximate or exact solutions for problems arising in physics, scientific and engineering applications.
4. The students will be able to choose the appropriate techniques from variational calculus to find e exact and qualitative solutions of differential equations.

2 Outline

1. Chapter ONE, INTRODUCTION: What will you get out of the course
2. Chapter TWO, FLOWS on the line: Solving
$$\dot{x}(t) = f(x(t)).$$
3. Chapter THREE, BIFURCATION: elementary bifurcation theory
4. Chapter FOUR, FLOWS on a circle
5. Chapter FIVE: LINEAR SYSTEMS
6. Chapter SIX: PHASE PLANE
7. Chapter SEVEN: LIMIT CYCLES
8. The End?
9. Chapter NINE:Lorenz equations
10. Chapter TEN: Maps, fractals, Strange attractors

BOOK S. H. Strogatz, Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering, Addison-Wesley. *Second Edition*

3 Grade Policy

Homework projects **30 %** + *two best Midterm Exams* **35 %** + *Final Exam+* **35**
Optional, subject to availability: Final Project +**10 %**
"Cell Phone" usage in class -5 percent of the final grade
I will assign a grade based on the following:
$90 \leq g \leq 100$ *is* A
$80 \leq g < 90$ *is* B
$70 \leq g < 80$ *is* C
$67 \leq g < 70$ *is* D
$g < 67$ *is* F
Grades modifiers will be used: $0 \leq g \leq 3$ *is "-",* $3 < g \leq 6$ *has no modifier,* $6 < g$ *is "+".*
There is no "make up" policy for homeworks.

Midterm Make-up Policy

If you miss one midterm, you will receive a grade of zero for that exam. This zero will be the grade that is removed from your overall score, as described in the grading policy.

If you miss a second midterm, the make-up will be in the form of an in-depth oral exam on the material covered. You will be randomly assigned one paragraph title to present. After *a preparation time of about 20 minutes, you will present the material of that paragraph to the professor. During the presentation, you will be asked questions to probe the depth and rigor of your understanding. You will be given a problem to solve.*

Home Works

ALL HOMEWORKS ARE REQUIRED

Late Homeworks: 1 % off per each full hour.

This policy is strictly enforced

I did hear stories about dogs eating homeworks.

Please submit HW via LMS

There is no "make up" policy for homeworks or exams.

Final Project

Time line for final project:

- **stage one** - *you think about what you want to do, talk to your friends, talk to me, bounce ideas of the wall - September 15, 2026*
- **stage 2**- *you submit to me a title and a one page description of your project September 30, 2026*
- **stage 3** *October-November: You work with me individually to conduct research and prepare presentation. This will typically involve anywhere from four to six meetings, where we discuss your research project and brainstorm to make it interesting. You prepare and practice with me your final talk.*
- **stage 4** - *you give me your talk November 15, 2025*
- **stage 4** *you give in-class presentation and answer questions December 1, 2026*
- *Last day of classes - you give me your written report in LATEX format*

To be eligible for final project your grade, with out the final project, should be A,B or C.

3.1 Few notes, policies, etc

There is no "curve grading".

Attendance *Attendance is strongly encouraged and may be logged. You are responsible for knowing everything that is said and discussed in class. The class webpage is provided as a resource, but relying on it alone is at your own risk.*

Use of Electronic Devices *The use of **laptops, cell phones, iPods, or lightsabers** during class is not permitted and will result in a 5% deduction from your final grade, unless the device is used specifically for taking notes.*

Please Submit your HW on LMS.

My promise: Those with passing grade will be proficient with Dynamical Systems.

Academic Integrity

The grade you receive for the course will be based on the work that you do.

With this principle in mind, the work (exams, homework, computer programs) that you present for a grade MUST, in fact, be of your own.

With respect to the exams, this means that no assistance or collaboration of any kind is permitted (other then assisace obtained from the instructor). Anyone violating this policy will receive an exam grade of zero and will be reported to the Dean of Students.

With respect to problem sets (homeworks and or computer projects), you are free to seek assistance or advise from any person, book or computer. However, what you hand in must be your own work. In this regard, **computer files must not be shared or exchanged** *nor should you copy work from someone else. If you do collaborate, please* **provide names** *of people you worked with at the beginning of your assignment. Violating of this policy will result in a score of zero for the assignment or the exam, and will be reported to the Dean of Students.*

Second violation will get you grade F for the course.

Note that your health, need of financial aid, need to maintain GPA, need to graduate, obtain employment, etc **will not** *be considered.*

BUT if you sense you have a problem, please talk to me. I may be able to help!

3.2 Important Note on HW's

It is your responsibility *to present your Homework in a best possible way. Please write neatly (better type), and respect grader/instructor time. Do not expect grader/instructor to figure out what you meant. Make it crystal clear!*

4 ODE Yoda wisdom

Curiosity is your superpower, questions set you free, and participation turns knowledge into your own.

- *Asking, the beginning of wisdom is.*
- *Sleeper you must not be. Awake, or lost you are.*
- *Silence feeds fear. Speak, and free yourself you will.*
- *The question unasked, the answer never found.*
- *Curiosity, your greatest ally it is.*
- *A mind that asks, alive it becomes. A mind that sleeps, withers it does.*
- *To learn, forget fear of being wrong you must.*

- *Ask you must, learn you will.*
- *Silent sit not, speak you must.*
- *Fear of wrong forget, question boldly you shall.*
- *Knowledge grow, questions bring it.*
- *Think deeply, speak freely, understand fully.*
- *Every question a spark is — light the fire you must.*
- *Learn you do, only when asking you are.*
- *Challenge assumptions, wisdom follow it will.*
- *Awake you must be, or lost your mind is.*

5 Lecture Notes Fall 2024

- **Lecture 1 August 29, 2024 Introduction, and Phase Plane technique:** $\dot{x}(t) = sin(x(t))$ **and** $\dot{x}(t) = 1 - x^2(t)$**.**
- **Lecture 2 September 6, 2024 Stability Analyses, Theorem of existence and uniqueness, impossibility of oscillations**
- **Lecture 3 September 10, 2024 Higher order stability analyses, potential and saddle node bifurcation**
- **Lecture 4 September 13, 2024 Normal form of Saddle Node Bifurcation, transcritical bifurcation, laser**
- **Lecture 5 September 17, 2024 Laser Threshold, pitchfork bifurcation, supercritical and subcritical**
- **Lecture 5 September 20, 2024 Subcritical bifurcation, hysteresis,overdamped bid on a hoop**
- **Lecture 6 September 24, 2024 ?**
- **Lecture 7 September 27, 2024 Review of Chapter 2 and 3. Motion on a circle.**
- **Lecture October 1, 2024 Midterm**
- **Lecture October 4, 2024 Nonuniform oscillator, pendulum with a torque**
- **Lecture October 8, 2024 Torque, Fireflies and beginning of** $\dot{x} = Ax$
- **Lecture October 11, 2024 Solving the matrix system** $\dot{x}(t) = Ax(t)$ **with matrix** A **and verctor** x

- **Lecture October 15, 2024 Solving $\dot{x}(t) = Ax(t)$ with complex eigenvalues/**
- **Lecture October 18, 2024 Deffective node and Classifications of the $\dot{x}(t) = Ax(t)$ systems**
- **Lecture October 22, 2024** *Classification of $\dot{x}(t) = Ax(t)$ and mathematics of love affairs http://www.yurilvov.com/ODElecture15.pdf*
- **Lecture October 25, 2024 System of nonlinear equations**

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}).$$

  Examples. Lotka-Valtera system
- **Lecture October 29, 2024 Midterm Review**
- **Lecture November 1, 2024 Midterm**
- **Lecture November 5, 2024 Lotka-Voltera and conservative systems**
- **Lecture November 8, 2024 Conservative and TIme Reversable systems**
- **Lecture November 12, 2024 Time Reversibility examples. Oscillator**
- **Lecture November 15, 2024 Oscillator and Limited Cycle. Van-der-Paul oscillator**
- **Lecture November 19, 2024 Lapunov, Dullac and Poincare-Bendixson**
- **Lecture November 20 (make up), 2024 Limited cycle, Lienard theorem and perturbation Theory**
- **Lecture November 22, 2024 Perturbation Theory Explained**
- **Lecture November 26, 2024 Perturbation Theory of the Van der Pol Oscillator**
- **Lecture December 3, 2024 ?**
- **Lecture December 6, 2024 no lecture**
- **Lecture December 10, 2024 ?**

6 Homeworks

6.1 Homework ONE

- *2.2.6*
- *2.2.8*
- *2.3.3*

- *2.4.8*
- *2.5.4*
- *2.6.1*
- *2.7.6*

6.2 Home Work TWO

- *3.1.4*
- *3.2.2*
- *3.3.2*
  In this problem, "l" stands for lower case "L" for "Larry", not "1" for "one".
- *3.4.4*
- *3.4.11*
- *3.4.12*
- *3.5.4 Write equation in a dimensional form first, and then derive the dimensionless form similar to what we did in lectures.*
- *3.7.6*

6.3 Home Work THREE

- *4.1.5*
- *4.1.9*
- *4.2.2*
- *4.3.4*
- *4.4.4*
- *4.5.1*

6.4 Home Work Four

- *5.2.1*
- *5.2.2*
- *5.2.6*
- *5.2.13*

- *5.3.3*
- *5.3.5*

6.5 Home Work Five

- *6.1.6*
- *6.2.1*
- *6.3.4*
- *6.4.3*
- *6.5.19*
- *6.6.3*
- *6.6.6. extra credit 15 percent*
- *6.7.2*

6.6 Home Work SIX

- *7.1.3*
- *7.2.1*
- *7.2.6*
- *7.3.3*
- *7.6.2*
- *7.6.3*

6.7 Home Work Seven

- *9.2.2 The smallest possible value of* C *is extra credit problem*
- *9.2.3*
- *9.2.4*
- *9.3.3 Extra Credit, 25 percent*
- *9.3.10 Extra Credit, 25 percent*

7 Example Problems

1. *Draw the phase portraits which has exactly three closed orbits and one fixed point.*
2. *Show that the system*
$$\dot{x} = y,\ \ \dot{y} = x\cos(y)$$
is reversible and sketch phase portrait

8 Examples

8.1 Trajectories of various equations

$\dot{x} = \sin x$

```
Show[Table[
Plot[Evaluate[x[t]/.NDSolve[{x'[t] == Sin[x[t]], x[0] == x0}, x,
{t, 0, 3}]],{t,0,3},PlotRange->{{0,3},{-10,10}},Frame->True],{x0,-3Pi,3Pi,.5}]]
```

Population $\dot{x} = x(1-x)$

```
Show[Table[
Plot[Evaluate[x[t]/.NDSolve[{x'[t] == x[t](1-x[t]), x[0] == x0}, x,
{t, 0, 3}]],{t,0,3},PlotRange->{{0,3},{0,4}},Frame->True],{x0,0,4,.25}]]
```

$\dot{x} = r - x - e^{-x}$

```
Show[Plot[x + Exp[-x],{x,-1,0},PlotRange-{{-1,1},{-1,1}}],
     Plot[x + Exp[-x],{x,0,1},PlotRange-{{-1,1},{-1,1}},
PlotStyle->Dashing[{0.1,0.1,.01}]]]
```

8.2 Modified Pitchfork bifurcation diagram of

$$\dot{x} = rx + x^3 - x^5$$

```
F1[r_] := Sqrt[1 - Sqrt[1 + 4 r]];
F2[r_] := Sqrt[1  Sqrt[1 + 4 r]];
Plot[{F1[r], -F1[r], F2[r], -F2[r], 0}, {r, -1, 2}]
```

8.3 Period of nonuniform oscillator by Jacob Harris

Show that $\int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin(\theta)} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$.

Let $u = \tan(\frac{\theta}{2})$.
$\rightarrow u(-\pi) = -\infty, u(\pi) = \infty$
$\rightarrow \theta = 2\arctan(u)$
$\rightarrow d\theta = \frac{2du}{1+u^2}$
$\rightarrow \tan(\theta) = \tan(2\arctan(u)) = \frac{2\tan(\arctan(u))}{1-\tan^2(\arctan(u))} = \frac{2u}{1-u^2} = \frac{\text{opp}}{\text{adj}}$
$\rightarrow \text{opp}^2 + \text{adj}^2 = 4u^2 + 1 - 2u^2 + u^4 = 1 + 2u^2 + u^4 = \text{hyp}^2$
$\rightarrow \text{hyp} = 1 + \text{u}^2$
$\rightarrow \sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{2u}{1+u^2}$
The integral becomes
$\int_{-\infty}^{\infty} \frac{2du}{(1+u^2)(\omega - a\frac{2u}{1+u^2})} = 2\int_{-\infty}^{\infty} \frac{du}{\omega + \omega u^2 - 2au}$
$= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{u^2 - \frac{2a}{\omega}u + 1}$
$= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{u^2 - \frac{2a}{\omega}u + \frac{a^2}{\omega^2} + 1 - \frac{a^2}{\omega^2}}$
$= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{(u-\frac{a}{\omega})^2 + 1 - \frac{a^2}{\omega^2}}$
Let $x = u - \frac{a}{\omega}$.
$\rightarrow dx = du$
$\rightarrow x(-\infty) = -\infty, x(\infty) = \infty$
The integral becomes
$\frac{2}{\omega}\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1 - \frac{a^2}{\omega^2}}$
Let $x = \sqrt{r}\tan(y)$, *with* $r = 1 - \frac{a^2}{\omega^2}$.
$\rightarrow y = \arctan(\frac{x}{\sqrt{r}})$.
$\rightarrow dx = \sqrt{r}\sec^2(y)dy$.
$\rightarrow y(-\infty) = -\pi/2, y(\infty) = \pi/2$
The integral becomes

$$\frac{2}{\omega}\int_{-\pi/2}^{\pi/2} \frac{\sqrt{r}\sec^2(y)dy}{r^2\tan(y)^2 + r^2}$$
$$= \frac{2}{\omega}\int_{-\pi/2}^{\pi/2} \frac{\sec^2(y)dy}{\sqrt{r}(\tan(y)^2 + 1)}$$
$$= \frac{2}{\omega}\int_{-\pi/2}^{\pi/2} \frac{\sec^2(y)dy}{\sqrt{r}\sec^2(y)}$$
$$= \frac{2}{\omega}\int_{-\pi/2}^{\pi/2} \frac{dy}{\sqrt{r}}$$
$$= \frac{2}{\omega\sqrt{1 - \frac{a^2}{\omega^2}}}(y)|_{-\pi/2}^{\pi/2}$$

$$= \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

8.4 Nonlinear Oscillator

```
Omega =1; a = 0.02;
       A = NDSolve[{Theta'[t] == Omega + a Sin[Theta[t]],
   Theta[0] == 0}, Theta, {t, 0, 200}];
Plot[Evaluate[Mod[Theta[t],2Pi] /. A], {t, 0, 100}]
```

8.5 Deffective Node

```
Animate[Show[StreamPlot[{-2 x + y, a y},
   {x, -3, 3}, {y, -3, 3}, PlotLabel -> {a}],
  Plot[{x /100, (a + 2) x}, {x, -3, 3}]],
 {a, -6, 6}]
```

8.6 Exaple 6.5.2

```
Plot3D[{0, y^2/2- x^2/2 +x^4/4},{x,-2,2},{y,-2,2}]
```

8.7 Van Der Pol Oscillator

```
StreamPlot[{v, -(x x - 1) v - x}, {x, -3, 3}, {v, -3, 3}]

A = NDSolve[{x''[t] + 3 (x[t]^2 - 1) x'[t] + x[t] == 0, x[0] == 3,
   x'[0] == 0}, x, {t, 0, 200}];
Plot[Evaluate[x[t] /. A], {t, 0, 30}]
```

8.8 Polar Stream Plot in Mathematica

Example of PolarStreamPlot in Mathematica

```
Needs["VectorAnalysis`"]
Clear[field, r, T, F];
m = Transpose[
     Transpose[JacobianMatrix[#, Spherical @@ #]]/
      ScaleFactors[Spherical @@ #]] &@{r, T, F};
```

```
field[r_, T_, F_] =
  Simplify[m.{3/(4 r^4) (3 Cos[T]^2 - 1),
     3/(4 r^4) Sin[2 T], 0}];

StreamPlot[

 Delete[

 field @@ CoordinatesFromCartesian[{x, 0, z}, Spherical], 2],

 {x, -3, 3}, {z, -3, 3}]
```

8.9 Perturbation Theory Example

```
epsilon = 0.3; Plot[{1/Sqrt[1 - epsilon^2] Exp[-epsilon t] Sin[
    Sqrt[1 - epsilon^2] t],
  Exp[-epsilon t] Sin[t]}, {t, 0, 20},
 PlotLegends -> {"Exact", "Pert"}, PlotRange -> {-0.5, 0.8}]
```

8.10 Lorenz

```
Sigma = 10;r=28;
solution = NDSolve[{
x'[t] == Sigma (y[t] - x[t]),
y'[t] == r x[t] - y[t] - x[t] z[t],
z'[t] == x[t] y[t] - 8/3 z[t], x[0] == z[0] == 0, y[0] == 2},
{x, y, z}, {t, 0, 35}];
ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. solution], {t, 0, 35}]
```

9 Past Exams and fun problems

1. *6.1.5 Consider the system*
$$\dot{x} = x(2 - x - y), \dot{y} = x - y.$$
Find the fixed points, determine their stability, sketch the nullclines, the vector field, and a plausible phase portrait.

2. *6.2.1 We learned in class that due to the uniqueness of a solution, the different trajectories can never intersect. But in many phase portraits, different trajectories appear to intersect at a fixed point. Is there a contradiction here? Explain this apparent contradiction.*

3. *6.3.5 Consider*
$$\dot{x} = \sin y, \dot{y} = \sin x.$$
Find the fixed points, classify them, sketch the neighboring trajectories, and fill in the rest of the phase portrait. Draw and use nulclines if it helps.

4. *6.3.9 Consider the system*
$$\dot{x} = y^3 - 4x, \dot{y} = y^3 - y - 3x.$$
   (a) *Find all the fixed points and classify them.*
   (b) *Show that the line $y = x$ is invariant, i.e., any trajectory that starts on it stays on it.*
   (c) *Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$. for all other trajectories. To achieve this form differential equation for $x - y$.*
   (d) *Sketch the phase portrait.*

5. *6.5.1*

   Consider the system
$$\ddot{x} = x^3 - x.$$
   (a) *Find all the equilibrium points and classify them.*
   (b) *Find a conserved quantity.*
   (c) *Sketch the phase portrait.*

6. *6.6.7 Consider*
$$\ddot{x} + x\dot{x} + x = 0.$$
Show that the system is time reversible and plot its phase portrait.

7. *Analyze the equation*
$$\dot{x} = 4x^2 - 16$$
graphically. Sketch the vector field on the real line, find all fixed points, classify their stabilityu and sketch the graph of $x(t)$ for different initial conditions. Obtain the solution in the closed form.

8. *Analyze the equation*
$$\dot{x} = 1 + \frac{1}{2}\cos(x)$$
graphically. Sketch the vector field on the real line, find all fixed points, classify their stabilityu and sketch the graph of $x(t)$ for different initial conditions.

   Extra Credit: Obtain the solution in the closed form.

9. *Consider the equation*
$$\dot{x}(t) = ax(t) - x^3(t).$$
Here a is a parameter that can be positive, negative or zero. Discuss all three cases.

Find all the fixed points of this equation and classify their stability by using the linear stability analyses. If the linear stability analyses fails, use the graphical method.

10. *2.5.3 Consider the equation*
$$\dot{x}(t) = rx(t) + x^3(t),$$
where $r > 0$ is a fixed parameter. Show that $x(t)$ goes to infinity in finite time, for any $x(t = 0) \neq 0$.

11. *Consider the equation*
$$\dot{x}(t) = 1 + rx + x^2.$$
Herr r is a parameter that can be positive, negative or zero. Plot all qualitatively different vector fields that occur as r varies. Show that a biffurcation occurs as r varies, and find the type of the biffurcation. Finally, plot the bifurcation diagram of x^* as a function of r.

12. *Consider the system*
$$\dot{x}(t) = rx + ax^2 - x^3.$$
Here both r and a are external parameters that can be positive, negative or zero. When $a = 0$ we have a normal form of the supercritical pitchfork.

(a) For each value of a there is a bifurcation diagram of x^* versus r. As a varies, the bifurcation diagram changes and goes through qualitatively different behaviour. Please plot all qualitatively different bifurcation diagrams that can be obtained as a varies.

(b) Summarize your finding by plotting the regions in the (a, r) plane that corresponds to qualitatively different classes of vector fields. Bifurcations occur on the boundaries of these regions. Identify the type of these bifurcations.

13. *Analyze the equation*
$$\dot{x} = e^x - \cos(x),$$
find all fixed points, and classify their stability.

HINT You do not need to find analytical solution for the fixed points

14. *For each of 1)-4) below, find an equation*
$$\dot{x} = f(x)$$
with that stated properties, or if there are no examples, exlain why not. (Assume $f(x)$ is a smooth continuous function).

   (a) *Every real number is a fixed point*
   (b) *There are precisely three fixed points, and all of them are stable*

(c) *There are no fixed points*

(d) *There are exactly 100 fixed points.*

15. *Show that the solution to*

$$\dot{x} = 1 + x^{10}$$

escapes to $+\infty$ *in a finite time, starting any initial condition.*

HINT Do not try to find an exact solution, instead, compare the solution to those of

$$\dot{x} = 1 + x^2.$$

16. *Consider the equation*

$$\dot{x} = r - x + x^3$$

for various values of r. *Plot the potential function* $V(x)$ *for* $r < 0$, $r = 0$ *and* $r > 0$, *and identify all equilibrium points and their stability.*

17. *Consider the equation*

$$\dot{x} = r - x + x^3$$

for various values of r. *Plot the bifurcation diagram of this equation and classify the bifurcation.*

18. *2.2.5*

Analyze the equation

$$\dot{x} = 1 + \frac{1}{2}\cos(x)$$

graphically. Sketch the vector field on the real line, find all fixed points, classify their stabilityu and sketch the graph of $x(t)$ *for different initial conditions.*

Extra Credit: Obtain the solution in the closed form.

19. *(The Allee effect) For certain species of organisms, the effective growth rate* $\dot{N}(t)/N$ *is highest at intermediate* N. *This is called the Allee effect (Edelstein–Keshet 1988). For example, imagine that it is too hard to find mates when* N *is very small, and there is too much competition for food and other resources when* N *is large.*

a) Show that

$$\frac{\dot{N}}{N} = r - a(N - b)^2$$

provides an example of the Allee effect, if r, a, *and* b *satisfy certain constraints, to be determined.*

b) Find all the fixed points of the system and classify their stability.

c) Sketch the solutions $N(t)$ *for different initial conditions.*

d) Compare the solutions $N(t)$ *to those found for the logistic equation. What are the qualitative differences, if any?*

20. 2.4.7 Consider the equation

$$\dot{x}(t) = ax(t) - x^3(t).$$

Here a is a parameter that can be positive, negative or zero. Discuss all three cases.

Find all the fixed points of this equation and classify their stability by using the linear stability analyses. If the linear stability analyses fails, use the graphical method.

21. 2.5.3 Consider the equation

$$\dot{x}(t) = rx(t) + x^3(t),$$

where $r > 0$ is a fixed parameter. Show that $x(t)$ goes to infinity in finite time, for any $x(t = 0) \neq 0$.

22. 3.1.1 Consider the equation

$$\dot{x}(t) = 1 + rx + x^2.$$

Herr r is a parameter that can be positive, negative or zero. Plot all qualitatively different vector fields that occur as r varies. Show that a biffurcation occurs as r varies, and find the type of the biffurcation. Finally, plot the bifurcation diagram of x^* as a function of r.

23. 3.6.3

Consider the system

$$\dot{x}(t) = rx + ax^2 - x^3.$$

Here both r and a are external parameters that can be positive, negative or zero. When $a = 0$ we have a normal form of the supercritical pitchfork.

(a) For each value of a there is a bifurcation diagram of x^* versus r. As a varies, the bifurcation diagram changes and goes through qualitatively different behaviour. Please plot all qualitatively different bifurcation diagrams that can be obtained as a varies.

(b) Summarize your finding by plotting the regions in the (a, r) plane that corresponds to qualitatively different classes of vector fields. Bifurcations occur on the boundaries of these regions. Identify the type of these bifurcations.

24. Sketch all the qualitatively different vector fields that occur as r is varied. Show a saddle-node bifurcation occurs at critical value of r to be determined. Finally, scetch the bifurcation diagram of fixed points x^* versus r:

$$\dot{x}(t) = 1 + rx + x^2.$$

25. Sketch all the qualitatively different vector fields that occur as r is varied. Scetch the bifurcation diagram of fixed points x^* versus r, and explain the bifurcation diagram.

$$\dot{x}(t) = r^2 - x^2.$$

26. *Sketch all the qualitatively different vector fields of the equation*

$$\dot{x} = x - rx(1 - x)$$

that occur as r *varied. Show that a transcritical bifurcation occurs at a critical value of* r *to be determined.*

27. *Find the values of* r *at which tbifurcations occur for the equation*

$$\dot{x} = 5 - re^{-x^2},$$

and classify those as saddle-node, transcritical, supecritical pitchfork or subcritical pitchfork. Sketch the bifurcation diagram of fixed points x^* *versus* r.

28. *Find and classify all fixed points of the following equation on a circle* $0 \leq \theta \leq 2Pi$*:*

$$\dot{\theta} = \sin^3 \theta,$$

29. *Draw the phase potraits of the following equation on a circle* $0 \leq \theta \leq 2Pi$*:*

$$\dot{\theta}(t) = \frac{\sin \theta(t)}{\mu + \sin \theta(t)},$$

as a function of control parameter μ*, classify the bifurcation that occurs as* μ *varies, and find bifurcation values of* μ.

30. *Consider the linear system*

$$\dot{x} = 3x - 4y, \quad \dot{y} = x - y.$$

Plot the phase portrait of this system and classify the fixed point. If the eigenvectors are real, plot them on your sketch.

31. *Consider the system*

$$\dot{x} = x(x - y), \quad \dot{y} = y(2x - y).$$

Find all fixed points. Sketch the nullclines, the vector field and plausible phase portrait.

32. *Consider the system*

$$\dot{x} = x - y, \quad \dot{y} = x^2 - 4.$$

Find the fixed points, classify them , sketch neighboring trajectories, and try to fill in the rest of the portrait.

33. *Consider the system*

$$\ddot{x} = x^3 - x.$$

- *Find all equilibrium points and classify them*
- *Find a conserved quantity*
- *Sketch the phase portrait*

34. *Analyze the equation*

$$\dot{x} = e^{-x}\sin x$$

graphically. Sketch vector field on the real line, find all the fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions.

35. *Find the values of r at which bifurcation(s) occur for the equation*

$$\dot{x} = rx - \frac{x}{1+x}$$

and classify them as saddle-node, transcritical or super(sub)critical pitchfork. Sketch the bifurcation diagram of fixed points x^* versus r.

36. *4.1.7 Find and classify all fixed points of*

$$\dot{\theta} = \sin k\theta,$$

where k is a positive integer. Sketch the phase portrait on a circle.

Two runners run a track in the opposite directions. One runner complete the one circle on a track in 2 minutes 30 seconds, another complete one cirle in 3 minutes 45 seconds. If they start running from the same starting point, how much time it would take them to meet again at the starting point? How many times they will meet in between start and finish?

37. *4.1.6 Consider the equation*

$$\dot{\theta}(t) = 3 + \cos(2\theta(t)).$$

Find all the fixed points, classify their stability, and plot the phase portraits on a cirle.

38. *4.3.5 Consider the system:*

$$\dot{\theta}(t) = \mu + \cos(\theta(t)) + \cos(2\theta(t)).$$

Draw the phase portraits as a function of a control parameter μ. Classify the biffurcations that occur as μ varies, find the values of μ where biffurcations occur.

39. *Consider a nonuniform oscillator*

$$\dot{\theta}(t) = \mu - \cos(\theta(x)).$$

Draw the phase portraits as a function of a control parameter μ. Classify the biffurcations that occur as μ varies, find the values of μ, where biffurcation occur, and plot the biffurcation diagram.

40. *Find the general Solution of*

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Plot the phase portrait and classify the stability of the fixed point(s). If the eigenvectors are real, plot them on your plot.

41. *Find the general solution of*

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Plot the phase portrait and classify the stability of the fixed point(s). If the eigenvectors are real, plot them on your plot.

42. *Find the general solution of*

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Plot the phase portrait and classify the stability of the fixed point(s). If the eigenvectors are real, plot them on your plot.

43. *Show that any matrix of the form*

$$\mathbf{A} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

has only onedimensional eigenspace corresponding to the eigenvalue λ.

Solve the system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} lambda & b \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

and plot the phase portrait.

44. *Consider the R and J interactions described by*

$$\frac{d}{dt}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix}.$$

Now answer the question:
Do the opposite attract? In other words, consider that J is an opposite of R, i.e.

$$c = -a, d = -b.$$

Now study the course of their interactions depending on relative sizes and signs of a and b. Draw characteristic phase portraits for all possible scenarios and describe by words interactions between R and J.

45. *Consider the system*

$$\dot{x} = 4x - y, \quad \dot{y} = 2x + y.$$

- *Find the general solution of the system*
- *Classify the fixed point at the origin*
- *Solve the system subject to initial conditions*

$$x(t = 0) = 3, y(t = 0) = 4.$$

46. 6.1.5 *Consider the system*

$$\dot{x} = x(2-x-y), \dot{y} = x-y.$$

Find the fixed points, determine their stability, sketch the nullclines, the vector field, and a plausible phase portrait.

47. 6.2.1 *We learned in class that due to the uniqueness of a solution, the different trajectories can never intersect. But in many phase portraits, different trajectories appear to intersect at a fixed point. Is there a contradiction here? Explain this apparent contradiction.*

48. 6.3.5 *Consider*

$$\dot{x} = \sin y, \dot{y} = \sin x.$$

Find the fixed points, classify them, sketch the neighboring trajectories, and fill in the rest of the phase portrait. Draw and use nulclines if it helps.

49. 6.3.9 *Consider the system*

$$\dot{x} = y^3 - 4x, \dot{y} = y^3 - y - 3x.$$

   (a) *Find all the fixed points and classify them.*

   (b) *Show that the line $y = x$ is invariant, i.e., any trajectory that starts on it stays on it.*

   (c) *Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$. for all other trajectories. To achieve this form differential equation for $x - y$.*

   (d) *Sketch the phase portrait.*

50. 6.5.1

Consider the system

$$\ddot{x} = x^3 - x.$$

   (a) *Find all the equilibrium points and classify them.*

   (b) *Find a conserved quantity.*

   (c) *Sketch the phase portrait.*

51. 6.6.7 *Consider*

$$\ddot{x} + x\dot{x} + x = 0.$$

Show that the system is time reversible and plot its phase portrait.

52. *Consider the*

$$\dot{x} = x - x^3, \quad \dot{y} = -y.$$

Find all fixed points. Sketch the nullclines, the vector field and plausible phase portrait.

53. *Consider the system*

$$\dot{x} = xy - 1, \quad \dot{y} = x - y^3.$$

Find the fixed points, classify them , sketch neighboring trajectories, and try to fill in the rest of the portrait.

54. *Consider the system*

$$\dot{x} = -y - x^3, \dot{y} = x.$$

Show that the origin is a spiral, although linearization predicts a center. HINT Think about nonlinear oscillator.

55. *Consider*

$$\ddot{x} + \alpha\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0, \quad \alpha > 0.$$

- *Find and classify all fixed points*
- *Show that the system has a circular limit cycle, and find its amplitude and period.*
- *Determine the stability of a limit cycle.*
- *Plot plausible phase portrait.*

56. *Find equilibrium points, classify their stability and draw representative $x(t)$ trajectories for*

$$\dot{x} = x(x-1)(x-2).$$

57. *Find Equilibrium points, classify their stability and draw representative trajectories and biffurcation diagram for the equation*

$$\dot{x}(t) = r(x^2(t) - r^2)x(t).$$

Please make sure to consider the cases $r > 0$, $r < 0$ and $r = 0$.

58. *2.5.1 A particle travels pn the half line $x \geq 0$with velocity given by*

$$\dot{x} = -x^c,$$

where c is real and constant.

(a) *Find all values of c such the origin $x = 0$ is stable fixed point*

(b)*Now assume that c is chosen so that the $x = 0$ is stable. Can the particle ever reach the origin in a finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?*

59. *Find the values of r at which biffurcation occur, and classify those as saddle point, transcritical or pitchfork biffurcation. Plot the biffurcation diagram.*

$$\dot{x} = x + \tanh(rx).$$

60. *Extra Credit Consider the equation*

$$\dot{x} = h + rx - x^2.$$

When $h = 0$m this system undergoes a transcritical biffurction at $r = 0$. Here we are going to study how the biffurcation diagram is affected by nonzero values of h.

(a) *Plot the biffurcation diagrams for $h = 0$, $h > 0$ and $h < 0$.*

(b) *Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields and identify the biffurcations that occur on the boundary of these regions.*

(c) *Plot the potential $V(x)$ that corresponds to all these different regions in the (r, h) plane.*

61. *Draw the phase portraits as a function of a control parameter μ. Classify biffurcation that occurs as μ varies and find all biffurcation values of μ:*

$$\dot{\theta}(t) = \mu \sin \theta(t) - \sin 2\theta(t).$$

62. *Find the conditions under which it is appropriate to approximate the equation*

$$mL^2\ddot{\theta}(t) + b\dot{\theta}(t) + mgL\sin(\theta(t)) = \mathcal{G}$$

by its overdamped limit

$$b\dot{\theta}(t) + mgL\sin(\theta(t)) = \mathcal{G}$$

63. *Find the general solution of the system*

$$\dot{x} = 2x - 2y, \quad \dot{y} = 2y - x.$$

Here $x \equiv x(t)$, $y \equiv y(t)$.

Sketch the vector field. Indicate the length and directions of the vectors with reasonalble accuracy. Sketch some typical trajectories.

64. *Plot the phase portrait and classify the fixed point of*

$$\dot{x} = 4x - 3y, \ \dot{y} = 8x - 6y.$$

If the eigenvectors are real, indicate them on your sketch.

65. *Predict the course of the events, depending on the signs and relative values of a and b.*

Do the opposites attract? Analyze

$$\dot{R}(t) = aR(t) + bJ(t), \quad \dot{J}(t) = -bR(t) - aJ(t).$$

This model would describe interaction of Fire and Water.

66. *Draw the phase portraits as a function of a control parameter μ. Classify biffurcation that occurs as μ varies and find all biffurcation values of μ:*

$$\dot{\theta}(t) = \mu \sin \theta(t) - \sin 2\theta(t).$$

67. *Find the conditions under which it is appropriate to approximate the equation*

$$mL^2\ddot{\theta}(t) + b\dot{\theta}(t) + mgL\sin(\theta(t)) = \mathcal{G}$$

by its overdamped limit

$$b\dot{\theta}(t) + mgL\sin(\theta(t)) = \mathcal{G}$$

68. *Find the general solution of the system*

$$\dot{x} = 2x - 2y, \quad \dot{y} = 2y - x.$$

Here $x \equiv x(t)$, $y \equiv y(t)$.

Sketch the vector field. Indicate the length and directions of the vectors with reasonalble accuracy. Sketch some typical trajectories.

69. *Plot the phase portrait and classify the fixed point of*

$$\dot{x} = 4x - 3y, \;\; \dot{y} = 8x - 6y.$$

If the eigenvectors are real, indicate them on your sketch.

70. *Predict the course of the events, depending on the signs and relative values of a and b. Do the opposites attract? Analyze*

$$\dot{R}(t) = aR(t) + bJ(t), \quad \dot{J}(t) = -bR(t) - aJ(t).$$

This model would describe interaction of Fire and Water.

71. *(The Allee effect) For certain species of organisms, the effective growth rate $\dot{N}(t)/N$ is highest at intermediate N . This is called the Allee effect (Edelstein–Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.*

a) Show that

$$\frac{\dot{N}}{N} = r - a(N-b)^2$$

provides an example of the Allee effect, if r, a, and b satisfy certain constraints, to be determined.

b) Find all the fixed points of the system and classify their stability.

c) Sketch the solutions $N(t)$ for different initial conditions.

d) Compare the solutions $N(t)$ to those found for the logistic equation. What are the qualitative differences, if any?

72. *Sketch the bifurcation diagram of the system*

$$\dot{x}(t) = x(t)\left(\mu - e^{x(t)}\right).$$

Determine what kind of bifurcations occur in this system, and at what values.

73. *Draw the phase portraits as a function of a control parameter μ. Classify biffurcation that occurs as μ varies and find all biffurcation values of μ:*

$$\dot{\theta}(t) = \frac{\sin(2\theta(t))}{1 + \mu\sin(\theta(t))}.$$

74. *Let A be a constant $n \times n$ matrix, and consider the system*

$$x\frac{d\mathbf{y}}{dx} = A\mathbf{y}, \qquad\qquad y \in \Re. \tag{1}$$

(i) Show that the substitution $x = e^t$ transforms (4) into a system with constant coefficients, and determine the coefficient matrix of that system.

(ii) Deduce that Euler's equation $x^2y'' + a_1xy' + a_0y = 0$ can be reduced to a second-order linear equation with constant coefficients via the substitution $x = e^t$.

(iv) Find the general solution of the system

$$x\frac{d\mathbf{y}}{dx} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{y}.$$

75. *Consider the system*

$$\dot{x} = xy - 1, \quad \dot{y} = x - y^3.$$

Find the fixed points, classify them , sketch neighboring trajectories, and try to fill in the rest of the portrait.

76. *Consider the planar system*

$$\dot{r}(t) = -r(t)(1 - r(t))(2 - r(t)), \qquad \dot{\theta}(t) = r^2(t),$$

which is expressed in terms of the polar coordinates $r(t)$ and $\theta(t)$.

- *Find the fixed point(s) of this system. What is the stability of this (these) fixed point(s)?*
- *Draw the phase portrait of this system in the $x - y$ plane, $x = r\cos\theta$, $y = r\sin\theta$.*
- *Does this system have periodic solutions? If yes, what are the periods of these periodic solutions?*

77. *Consider*

$$\ddot{x} + \alpha\dot{x}(x^2 + \dot{x}^2 - 3) + x = 0, \quad \alpha > 0.$$

- *Find and classify all fixed points*
- *Show that the system has a circular limit cycle, and find its amplitude and period.*
- *Determine the stability of a limit cycle.*
- *Plot plausible phase portrait.*

78. **Weakly nonlinear oscillations:** *Find the dependence of the period of oscillations of a pendulum described by the equation*

$$\ddot{x}(t) + x(t) + \epsilon\dot{x}^3(t) = 0 \tag{2}$$

on the amplitude of oscillations A, assuming that both ϵ and A are small. Develop a perturbation theory and multiple time scales to remove secular terms. What can you say about this system?

Note that

$$[A\sin(t) + B\cos(t)]^3 =$$
$$\frac{3B}{4}\cos(t)\left(A^2 + B^2\right) - \frac{B}{4}\left(3 * A^2 + B^2\right)\cos(3t) + \frac{3A}{4}\sin(t)\left(A^2 + B^2\right) - \frac{A}{4}\sin(3t)\left(A^2 + B^2\right).$$

Also,

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$
$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B)),$$
$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B)),$$

79. *Find equilibrium points, classify their stability and draw representative $x(t)$ trajectories for*

$$\dot{x} = \sin(x)(x-1)(x-2).$$

80. *Consider*

$$\frac{d}{dt}x(t) = f(x(t)).$$

Use the existence of the potential

$$\frac{d}{dt}x(t) = -\frac{dV(x(t))}{dx}$$

to show that $x(t)$ can not oscillate.

81. *Find Equilibrium points, classify their stability and draw representative trajectories and biffurcation diagram for the equation*

$$\dot{x}(t) = r(x^2(t) - r^2)x(t).$$

Please make sure to consider the cases $r > 0$, $r < 0$ and $r = 0$.

82. *A particle travels pn the half line $x \geq 0$with velocity given by*

$$\dot{x} = -x^c,$$

where c is real and constant.

(a) *Find all values of c such the origin $x = 0$ is stable fixed point*

(b)*Now assume that c is chosen so that the $x = 0$ is stable. Can the particle ever reach the origin in a finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?*

83. *Find the values of r at which biffurcation occur, and classify those as saddle point, trans-critical or pitchfork biffurcation. Plot the biffurcation diagram.*

$$\dot{x} = rx + x^3/(1+x^2).$$

84. *Extra Credit Consider the equation*

$$\dot{x} = h + rx - x^2.$$

When $h = 0m$ this system undergoes a transcritical biffurction at $r = 0$. Here we are going to study how the biffurcation diagram is affected by nonzero values of h.

(a) *Plot the biffurcation diagrams for $h = 0$, $h > 0$ and $h < 0$.*

(b) *Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields and identify the biffurcations that occur on the boundary of these regions.*

(c) *Plot the potential $V(x)$ that corresponds to all these different regions in the (r, h) plane.*

85. *Consider*

$$\dot{\theta}(t) = \frac{\sin\theta}{\mu + \sin\theta}.$$

Draw the phase portrait as a function of of the control parameter μ. Classify the biffurcation that occur as μ varies, and find all the biffurcation values of μ. Plot the biffurcation diagram.

86. *Consider the motion on a circle defined by the equation*

$$\dot{\theta}(t) = \frac{1}{\cos(\theta)}.$$

Plot the (circle) phase diagram for this equation, and few characteristic $\theta(t)$ curves.

Now consider the initial condition

$$\theta(t = 0) = \frac{3}{2}\pi - \epsilon,$$

where ϵ is extremely small positive number, say $\epsilon = 10^{-100}$. Then how much time would it take for the system to reach $\theta = \frac{\pi}{2}$?

87. *Find the general solution and plot the accurate phase portrait for the system*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \mathbf{x}(t).$$

Classify the stability of the origin. If the eigenvectors are real, plot them on your phase portrait.

88. *Find the general solution and plot the accurate phase portrait for the system*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 \\ -9 & 5 \end{bmatrix} \mathbf{x}(t).$$

Classify the stability of the origin. If the eigenvectors are real, plot them on your phase portrait.

89. *Romeo is the robot. Nothing coud ever change the way Romeo feels about Juliet:*

$$\dot{R} = 0, \dot{J} = aR + bJ.$$

Does Juliet end up loving him or hating him?

Under what conditions Juliet ends up loving Romeo?

90. *Consider the system*

$$\dot{x} = y^3 - 4x, \dot{y} = y^3 - y - 3x.$$

(a) *Find all the fixed points and classify their stability*

(b) *Show that the line $y = x$ is invariant, i.e. any trajectory which starts on this line stays on this line.*

(c) *Show that*

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$

for all other trajectories.
HINT: Form the ODE for $x(t) - y(t)$.

(d) *Sketch the phase portrait*

91. *Consider the system*

$$\dot{x} = -y - x^2, \dot{y} = x.$$

Show that the origin is a spiral, although the linearization predicts a center.

92. *Consider the system*

$$\ddot{x} = x^3 - x.$$

(a) *Find all fixed points and classify their stability*

(b) *Find a conserved quantity*

(c) *Does this system has periodic trajectories?*

(d) *Sketch the phase portrait*

93. *Consider the system*

$$\dot{x} = f(x, y), \dot{y} = g(x, y). \tag{3}$$

Let f and g be a smooth vector field defined on the phase plane.

(a) *Show that if*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x},$$

then the system (3) is a gradient system.

(b) *Is the previous condition sufficient for the (3) to be a gradient system?*

94. Show that the system

$$\ddot{x} + \mu(x^2 - 1)x + \tanh(x) = 0, \quad \mu > 0$$

has exactly one periodic solution.

95. Consider the Duffin oscillator, described by the differential equation

$$\ddot{x}(t) + x(t) + (x(t))^3 = 0.$$

- *Find the equation for the closed orbit* $\dot{x}(t) = g(x(t))$.
- *Find the period of this oscillator, if the initial conditions are given by*

$$x(t=0) = 1, \dot{x}(t=0) = 0.$$

96. Plot the bifurcation diagram of the system

$$\dot{x}(t) = \mu + x\sin(x).$$

Determine what kind of bifurcations occur in this system, and at what value(s).

Hint To gain intuition for this problem, plot $x\sin(x)$ *and move the ruller up and down the graph.*

97. Draw the phase portraits as a function of a control parameter μ. *Classify biffurcation that occurs as* μ *varies and find all biffurcation values of* μ:

$$\dot{\theta}(t) = \mu + \cos(\theta) + \cos(2\theta).$$

98. Let A *be a constant* $n \times n$ *matrix, and consider the system*

$$x\frac{d\mathbf{y}}{dx} = A\mathbf{y}, \qquad y \in \Re. \tag{4}$$

(i) Show that the substitution $x = e^t$ *transforms (4) into a system with constant coefficients, and determine the coefficient matrix of that system.*

(ii) Find the general solution of the system

$$x\frac{d\mathbf{y}}{dx} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{y}.$$

99. Consider the system

$$\dot{x} = \sin(y), \quad \dot{y} = \cos(x).$$

Find the fixed points, classify them , sketch neighboring trajectories, and fill in the rest of the portrait.

100. *Vasquez and Redner (2004, p. 8489) mention a highly simplified model of political opinion dynamics consisting of a population of leftists, rightists, and centrists. The leftists and rightists never talk to each other; they are too far apart politically to even begin a dialogue. But they do talk to the centrists. This is how opinion change occurs. Whenever an extremist of either type talks with a centrist, one of them convinces the other to change his or her mind, with the winner depending on the sign of the parameter r. If $r > 0$ the extremist always wins and persuades the centrist to move to that end of the spectrum. If $r < 0$ the centrists always wins and pulls the extremist to the middle. The model's governing equations are*

$$\dot{x} = rxz, \quad \dot{y} = ryz, \quad \dot{z} = -rxz - ryz,$$

where x, y, and z are the relative fractions of rightists, leftists, and centrists, respectively, in the population.

(a) Show that the set $x + y + z = 1$ is invariant. What does this invariant represent in the context of the model?

(b) Use the invariant to reduce this to a two variable system from the three variable system.

(c) Analyze the long term behavior predicted by the model for both positive and negative values of r.

(d) Interpret the results in political terms

(e) Propose a more realistic model

101. *Consider the competition model*

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2, \dot{N}_2 = r_2 N_2(1 - N_2/K_2 2) - b_2 N_1 N_2.$$

Useing the Dulac's criterion with the weighting function

$$g = \frac{1}{N_1 N_2},$$

show that the system has no periodic orbits in the first quadrant $N_1, N_2 > 0$.

102. *Consider the planar system*

$$\dot{r}(t) = -r(t)(1 - r(t))(2 - r(t)), \qquad \dot{\theta}(t) = r^2(t),$$

which is expressed in terms of the polar coordinates $r(t)$ and $\theta(t)$.

- *Find the fixed point(s) of this system. What is the stability of this (these) fixed point(s)?*
- *Draw the phase portrait of this system in the $x - y$ plane, $x = r\cos\theta$, $y = r\sin\theta$.*
- *Does this system have periodic solutions? If yes, what are the periods of these periodic solutions?*

Bonus Problem *Consider*

$$\ddot{x} + \alpha\dot{x}(x^2 + \dot{x}^2 - 3) + x = 0, \quad \alpha > 0.$$

- *Find and classify all fixed points*
- *Show that the system has a circular limit cycle, and find its amplitude and period.*
- *Determine the stability of a limit cycle.*
- *Plot plausible phase portrait.*

Bonus Problem

Weakly nonlinear oscillations: *Find the dependence of the period of oscillations of a pendulum described by the equation*

$$\ddot{x}(t) + x(t) + \epsilon\dot{x}^3(t) = 0 \tag{5}$$

on the amplitude of oscillations A, assuming that both ϵ and A are small. Develop a perturbation theory and multiple time scales to remove secular terms. What can you say about this system?

Note that

$$[A\sin(t) + B\cos(t)]^3 =$$

$$\frac{3B}{4}\cos(t)\left(A^2+B^2\right) - \frac{B}{4}\left(3*A^2+B^2\right)\cos(3t) + \frac{3A}{4}\sin(t)\left(A^2+B^2\right) - \frac{A}{4}\sin(3t)\left(A^2+B^2\right).$$

Also,

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$
$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B)),$$
$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B)),$$

(The Allee effect) For certain species of organisms, the effective growth rate $\dot{N}(t)/N$ is highest at intermediate N . This is called the Allee effect (Edelstein–Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

a) Show that

$$\frac{\dot{N}}{N} = r - a(N-b)^2$$

provides an example of the Allee effect, if r, a, and b satisfy certain constraints, to be determined.

b) Find all the fixed points of the system and classify their stability.

c) Sketch the solutions $N(t)$ for different initial conditions.

d) Compare the solutions $N(t)$ to those found for the logistic equation. What are the qualitative differences, if any?

ANSWER *Equilibria are $N = 0$ and $N = b \pm \sqrt{\frac{r}{a}}$. For $N \ll 1$, $\dot{N} = N(r - ab^2)$ so that the rate is $r - ab^2$. If $r < ab^2$ the rate is negative. Constraints are*

$$a > 0, r > 0, b^2 > r/a.$$

The difference with Logistics equation that rate may be negative, so small population decreases to zero. Zero is unstable Fixed Point