10 am, Tu   11-20 am Tu   1

1 sided crib sheet

Classification of differential equations

- ODE
  ordinary differential equations
  1 dependent variable
  time t,
  coordinate x

PDE
partial differential equations
x, t

- Equations vs systems
  1 dependent
  2 or more dependent variables

- Order of the differential equation - 3

  order of highest derivative

  1st order - I midterm

  2nd order

- Linear versus nonlinear

  dependent variable is in first order

  -//- is NOT -//-

- Linear ODE

  homogeneous $y(x) = 0$ is a solution

  unhomogeneous -//- NOT -//-

Separation of variables

ODE's $\frac{d}{dx} y(x) = \underbrace{F(x)\,G(y(x))}_{\text{separable}};$ Graphical calculator

$$\int \frac{d\,y(x)}{G(y(x))} = \int F(x)\,dx$$ Implicit solution

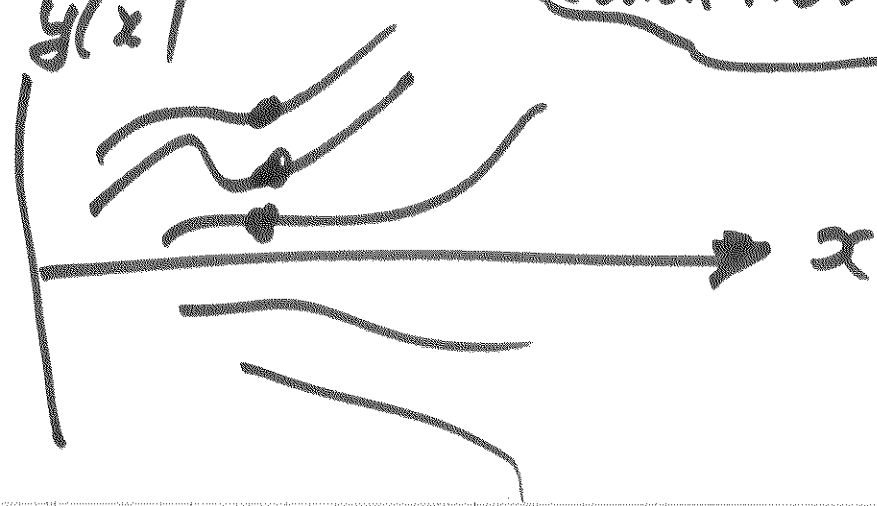
Explicit solution $y(x) = \dots$

One initial condition

ODE has a family of solutions

1st order

1 free constant



Integrating factor

$$\left[\frac{d\,y(x)}{dx} + p(x)y(x) = g(x);\right] e^{\int p(x)dx} = \mu(x)$$

$$e^{\int p(x)dx}$$

$$\underbrace{\frac{dy(x)}{dx} + p(x)\,y(x)\,e^{\int p(x)dx}}$$

$$= g(x)e^{\int p(x)dx}$$

$$\frac{d}{dx}\left(y(x)e^{\int p(x)dx}\right) = g(x)e^{\int p(x)dx}$$

$$y(x) = \frac{1}{\mu(x)}\int g(x)\mu(x)dx;$$

Modeling

– $N(t)$

↳ quantity of interest

$$\frac{dN(t)}{dt} = \mathrm{In}(N(t), t) - \mathrm{Out}(N(t), t)$$

Population growth

Simple: $\frac{dN(t)}{dt} = k N(t);\ k > 0$

$$N(t) = N(t=0)\, e^{kt}$$

- Given rate: when population increase factor of m

- Given facts about population increase
  IF doubles in T

$$N(t=T) = N(t=\emptyset)\, e^{k \cdot T} = 2N(t=\emptyset)$$

$$kT = \ln 2 \rightarrow k = \frac{1}{T}\ln 2$$

$$T = \frac{1}{k}\ln 2$$

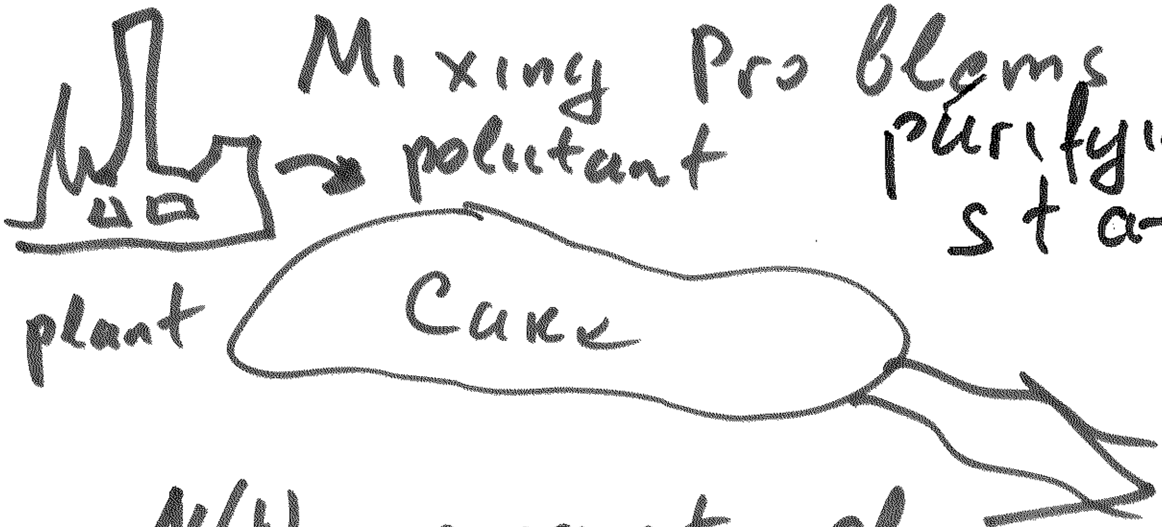
Logistic equation:

K - carrying capacity

$$\frac{d}{dt} N(t) = k \cdot N(t)\left(1 - \frac{N(t)}{K}\right);$$

$$\frac{d}{dt} N(t) = k\, N(t)\left(1 - \frac{N(t)}{K}\right) + \underset{\substack{\uparrow \\ \text{influx} \\ \text{of} \\ \text{people}}}{F(t)}$$

Mixing Problems



purifying station $\equiv$ Treatment plant

$N(t)$ amount of pollutant

$$\frac{d}{dt}N(t) = \varphi - \frac{N(t)}{v}\varphi - \overbrace{\frac{N(t)}{v}\cdot 100\varphi}^{\theta(t-T)}$$

$$\theta(t) = \begin{cases} \emptyset, & t \leq 0 \\ 1 & t > \emptyset \end{cases}$$

In

theta function

v - volume of a lake

- Initially, $t = \emptyset$, $N(t=\emptyset) = N_0$

IVP

initial value problem

$$\frac{d}{dt} N(t) = \varphi - \frac{N(t)}{v} \cdot \varphi$$

Obtain $N(t=T) = \tilde{N}$

- Treatment plant

treatment

$$\frac{d}{dt} N(t) = \frac{\varphi}{10} - \frac{N(t)}{v} \cdot \varphi - \frac{N(t)}{v} \cdot 100\varphi$$

decrease in polluting

Newton law: $F = ma = m \frac{d^2x(t)}{dt^2}$

$$v(t) = \frac{dx(t)}{dt}$$

$$\rightarrow F = m \frac{dv(t)}{dt} \Rightarrow v(t) = \ldots$$

$$\rightarrow \frac{dx(t)}{dt} = v(t) \Rightarrow x(t)$$

Newton law of cooling

$T(t)$ - temperature

$$\frac{d}{dt} T(t) = -k \left(T(t) - T_{AMBIENT} \right)$$

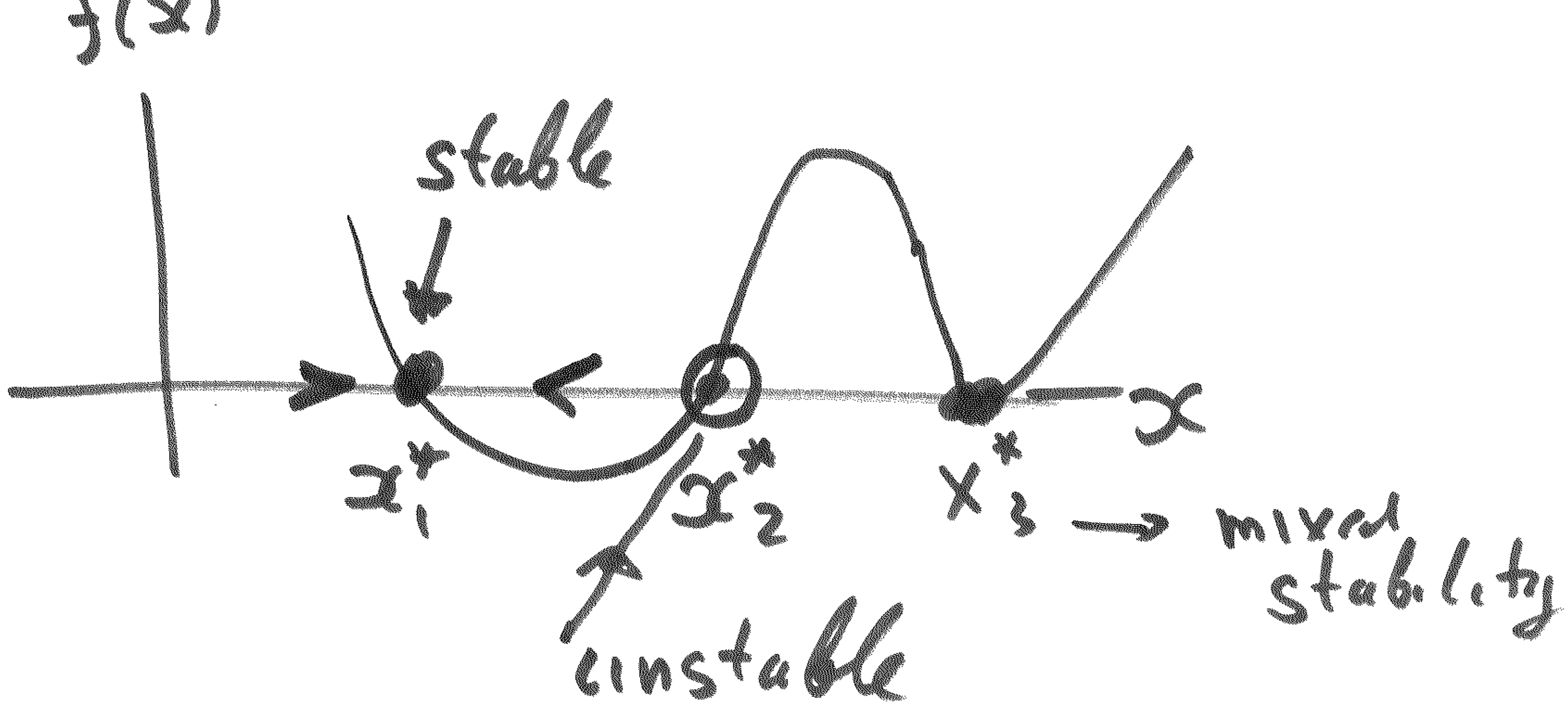
$k \rightarrow \frac{1}{\text{Sweater Thickness}}$

Nonlinear cooling

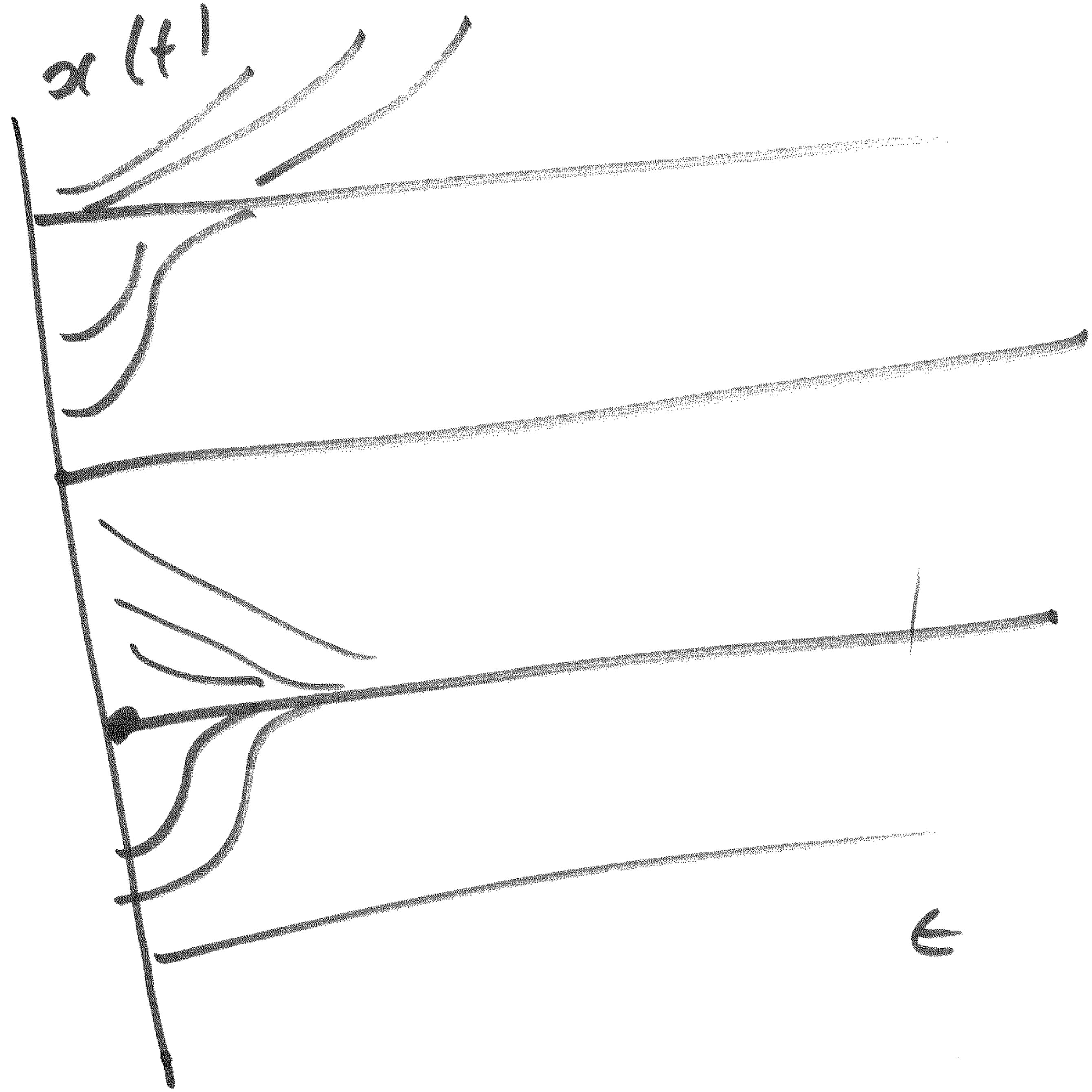
$\frac{d}{dt}x(t) = f(x(t))$ autonomous equation

Steady state, equilibrium, fixed point, critical point

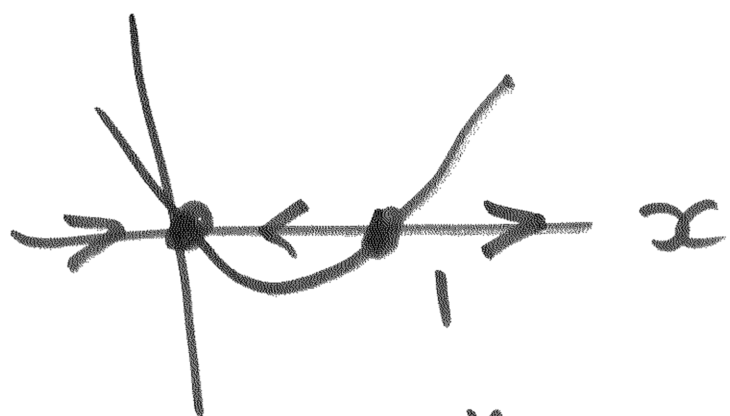
x^* such that $f(x^*) = 0$



24



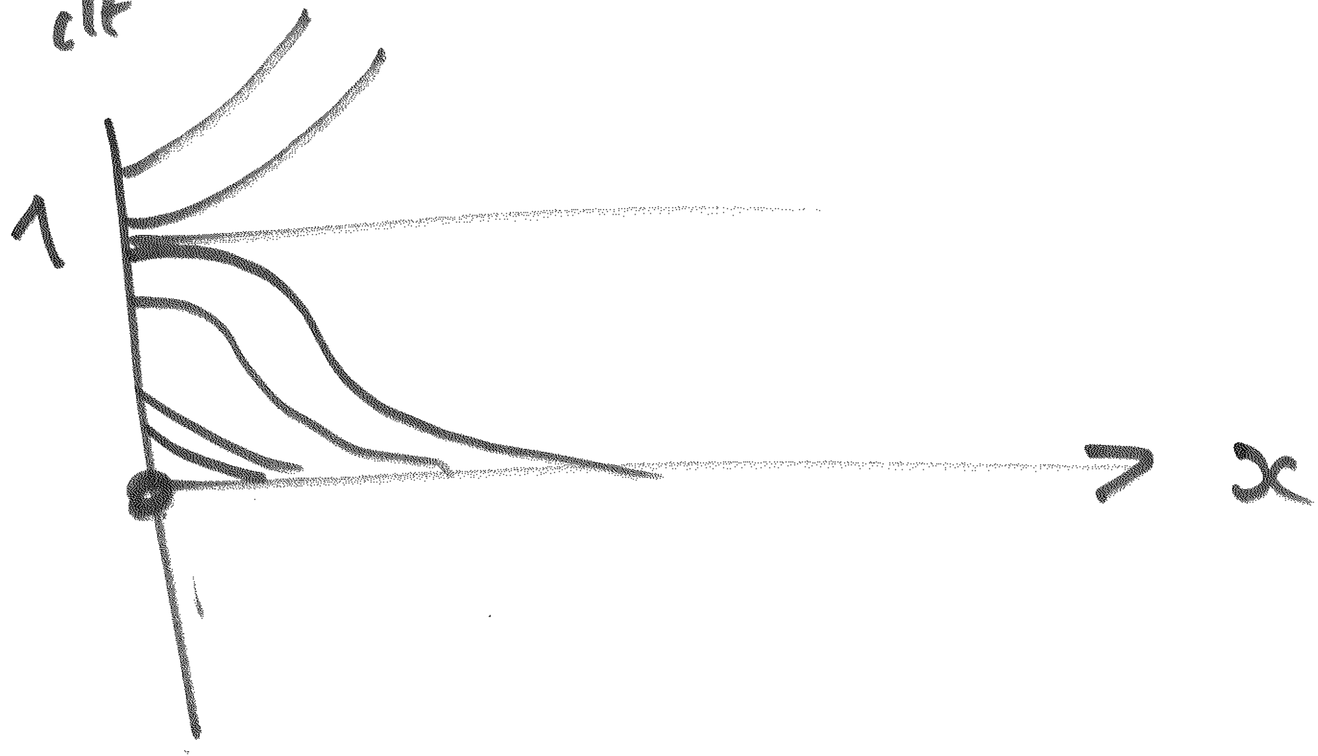
$$\frac{d}{dt}x(t) = -x(t)(1-x(t)); \quad x=0, \; x=1$$



$x=0$ - stable

$x=1$ unstable

$$\frac{d}{dt} x = x(1-x)$$



2nd order ode with constant coefficients:

$$a y''(x) + b y'(x) + c y(x) = \emptyset$$

$$y(x) = e^{rx};$$

$$ar^2 + br + c = \emptyset.$$

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 > 4ac$
- $b^2 = 4ac$
- $b^2 < 4ac$

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

29

$\Rightarrow b^2 > 4ac$, 2 real distinct roots

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$\Rightarrow b^2 = 4ac$; $y(x) = C_1 x e^{rx} + C_2 e^{rx}$

$r = r_1 = r_2$

$\Rightarrow b^2 < 4ac$

two complex conjugate roots $\quad 4ac > b^2$

$$r_1 = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

$$r_2 = \frac{-b + i\sqrt{4ac - b^2}}{2a}$$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$ — 1st form

$$y(x) = c_1 e^{x \operatorname{Re} r} \cos((\operatorname{Im} r_1) x) + c_2 e^{x \operatorname{Re} r}$$