Second order ODE's

$$y''(x) = F(x, y(x), y'(x));$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_0'$$

Linear Second ODE

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x) = F(x);$$

Homogeneous:

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x) = 0 \qquad (*)$$

IF $y_1(x)$ and $y_2(x)$ are solutions of $(*)$

Then So is $y(x) = C_1 y_1(x) + C_2 y_2(x)$

$y_1(x)$ is a solution

$$y_1''(x) + p(x)\,y_1'(x) + g(x)\,y_1(x) = \emptyset; \quad \times C_1$$

$y_2(x)$ is a solution

$$y_2''(x) + p(x)\,y_2(x) + g(x)\,y_2(x) = \emptyset; \quad \times C_2$$

$+$

$$C_1 y_1''(x) + C_2 y_2''(x) + C_1\,p(x)\,y_1'(x) + C_2\,p(x)\,y_2'(x) + C_1\,g(x)\,y_1(x) + C_2\,g(x)\,y_2(x) = \emptyset;$$

$$\frac{d^2}{dx^2}\overbrace{\left(C_1 y_1(x) + C_2 y_2(x)\right)}^{y(x)} + \underbrace{p(x)\frac{d}{dx}}_{y(x}\overbrace{\left(C_1 y_1(x) + C_2 y_2(x)\right)}^{y(x)} + g(x)\left(C_1 y_1(x) + C_2 y_2(x)\right) = \emptyset$$

Let $y_1(x)$ and $y_2(x)$ be solutions of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

and

$$y(x = x_0) = y_0$$

and

$$y'(x = x_0) = y_0'$$

Find C_1 and C_2 to satisfy IVP

Let $y(x) = C_1 y_1(x) + C_2 y(x)$;

$y(x_0) = y_0$: $\quad C_1 y_1(x = x_0) + C_2 y_2(x = x_0) = y_0$

$y'(x_0) = y_0$: $\quad C_1 y_1'(x = x_0) + C_2 y_2'(x = x_0) = y_0'$

$$\begin{pmatrix} y_1(x=x_0) & y_2(x=x_0) \\ y_1'(x=x_0) & y_2'(x=x_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Wronskian (1776-1853) 1812(?)

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

$Ax = b$,

$\mathrm{Det}(A) \neq 0 \Rightarrow W(x_0) \neq 0$

- Functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$C_1 y_1(x) + C_2 y_2(x) = \emptyset$$

$$\Updownarrow$$

$$C_1 = C_2 = \emptyset$$

- Functions $y_1(x), y_2(x) .. y_n(x)$ are linearly independent IFF

$$\sum_{k=1}^{n} C_k y_k(x) = \emptyset \iff C_1 = .. C_n = \emptyset$$

Examples

- $y_1(x) = x$, $y_2(x) = x^2$ are linearly independent

- $y_1(x) = 2x$; $y_2(x) = 3x$
  are linearly dependent

Homogeneous second order ODE with constants coefficients

$P(x) = b$; $q(x) = c$; b, c are given real constants

$y''(x) + b\, y'(x) + c\, y(x) = 0$;

$\rightarrow y(x) = e^x$; $y'(x) = e^x$; $y''(x) = e^x$;

$e^x + b e^x + c e^x = 0$

or $e^x (1 + b + c) = 0$

$\rightarrow y(x) = e^{rx}$;

$$y''(x) + b\,y'(x) + c\,y(x) = \emptyset;$$

$$y(x) = e^{rx};$$

$$y'(x) = r e^{rx}$$

$$y''(x) = r^2 e^{rx}$$

$$r^2 e^{rx} + b r e^{rx} + c e^{rx} = \emptyset$$

or $e^{rx}(r^2 + br + c) = \emptyset$

IF $x \neq \pm\infty$: $e^{rx} \neq \emptyset$

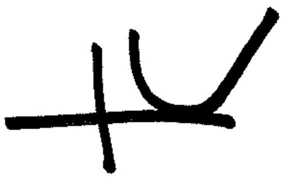
$$\boxed{r^2 + br + c = \emptyset}$$ Characteristic equation

$$r^2 + br + c = \emptyset;$$

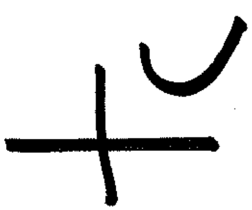
$$r^2 + 2\frac{b}{2}r + c = \emptyset$$

$$r_{12} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

- $b^2 > 4c$: two distinct real roots



- $b^2 = 4c$: one double real root



- $b^2 < 4c$: two complex conjugate roots

$$y''(x) + b\,y'(x) + c\,y(x) \qquad y(x) = e^{rx}$$

with $b^2 > 4c$

$$r_1 = \frac{1}{2}\left(-b + \sqrt{b^2 - 4ac}\right)$$

$$r_2 = \frac{1}{2}\left(-b - \sqrt{b^2 - 4ac}\right)$$

$$y_1(x) = e^{r_1 x}$$

$$y_2(x) = e^{r_2 x}$$

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

$$W(x) = \mathrm{Det}\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} =$$

$$= y_1(x)y_2'(x) - y_2(x)y_1'(x) = e^{(r_1+r_2)x}(r_2 - r_1) \neq 0$$

$y''(x) - y(x) = 0$

$y(0) = 3 \;;\; y'(0) = 1$

$y(x) = e^{rx}$;

$r^2 - 1 = 0 \Rightarrow r = \pm 1$

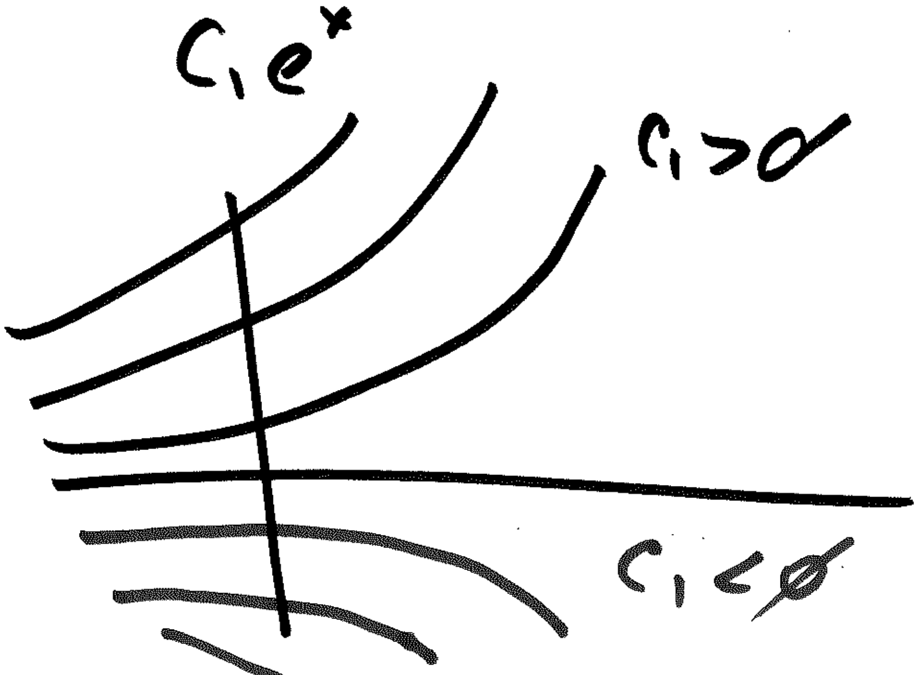
$y(x) = c_1 e^{x} + c_2 e^{-x}$

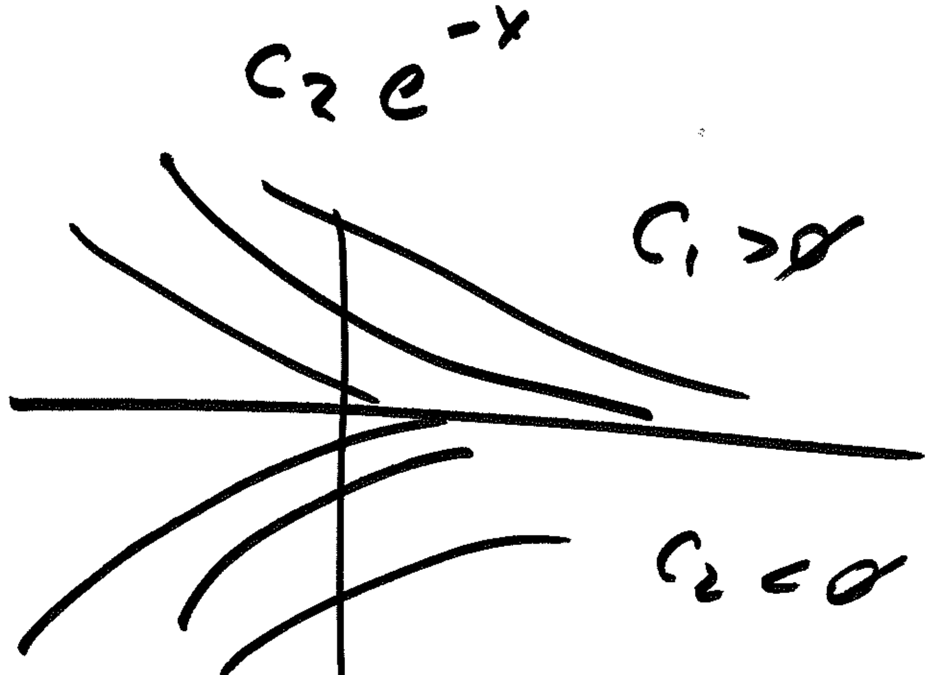
$y'(x) = c_1 e^{x} - c_2 e^{-x}$

$y(0) = \boxed{c_1 + c_2 = 3}$

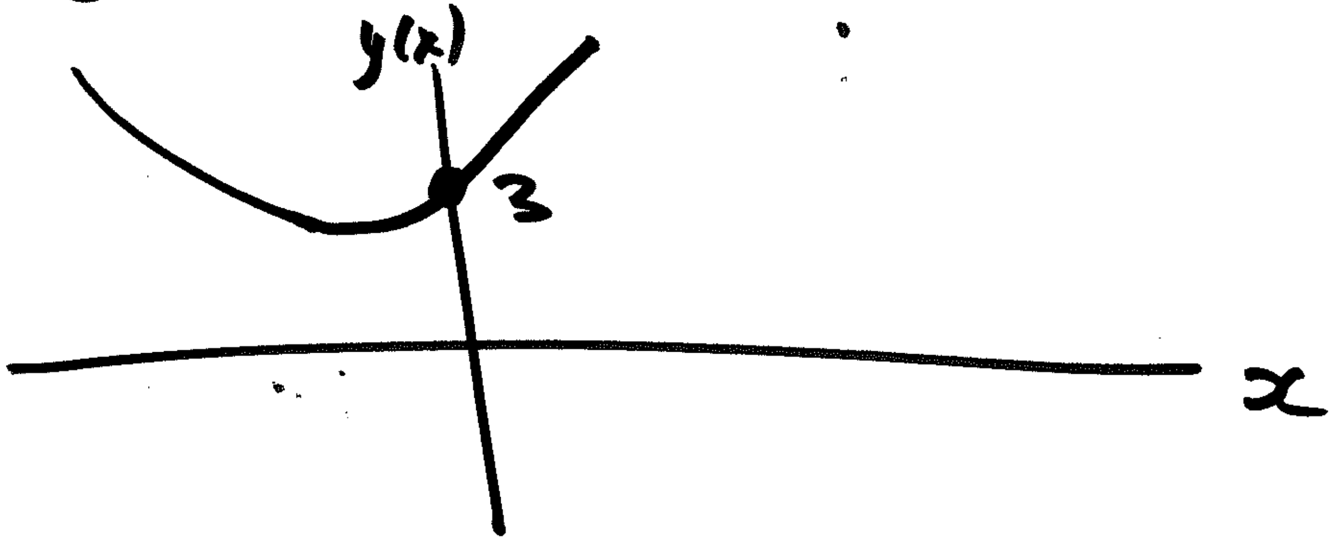
$y'(0) = \boxed{c_1 - c_2 = 1}$

$c_1 = 2;\; c_2 = 1$





$y(x) = 2e^{x} + e^{-x}$



$y''(x) + 5y'(x) + 6y(x) = 0$

$y(0) = 2; \quad y'(0) = 3;$

$y(x) = e^{rx};$

$y'(x) = re^{rx}$

$y''(x) = r^2 e^{rx}$

$r^2 e^{rx} + 5 \cdot r e^{rx} + 6 e^{rx} = 0; \quad e^{rx} \neq 0$

$r^2 + 5r + 6 = 0;$

$$r_{12} = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm 1}{2}$$

$$= -3; -2;$$

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$$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y'(x) = -2c_1 e^{-2x} - 3c_2 e^{-3x}$$

$$y(0) = c_1 + c_2 = 2$$

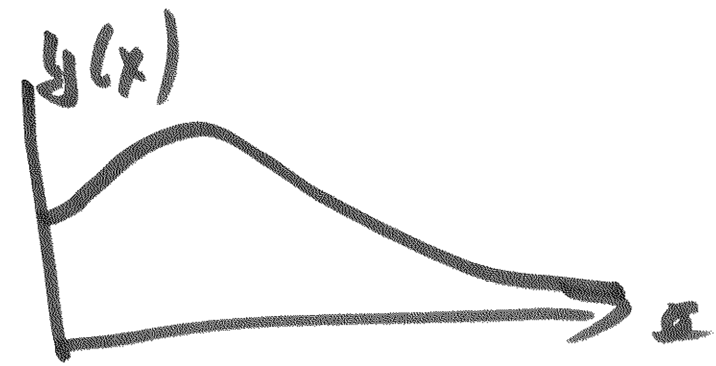
$$y'(0) = -2c_1 - 3c_2 = 3$$

$$c_2 = 2 - c_1; \quad -2c_1 - 6 + 3c_1 = 3$$

$$c_1 = 9$$

$$c_2 = 2 - c_1 = -7$$

$$y(x) = 9e^{-2x} - 7e^{-3x}$$



4 $y''(x) - 8y'(x) + 3y(x) = \emptyset;$

$y(\emptyset) = 2$

$y'(\emptyset) = 1/2$

$4r^2 - 8r + 3 = \emptyset;$

$$r_{12} = \frac{8 \pm \sqrt{64 - 48}}{8}$$

$$= \frac{8 \pm 4}{8} = \frac{1}{2};\ 3/2$$

$$y(x) = c_1 e^{x/2} + c_2 e^{3x/2}$$

$$y'(x) = \frac{c_1}{2} e^{x/2} + \frac{3}{2} c_2 e^{3x/2}$$

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = \frac{c_1}{2} + \frac{3}{2} c_2 = 1/2$$

$$c_1 = 5/2; \quad c_1 = -1/2;$$

$$y(x) = \frac{5}{2} e^{x/2} - \frac{1}{2} e^{3x/2};$$

$$c_2 = 2 - c_1$$

$$\frac{c_1}{2} + \frac{3}{2}(2 - c_1) = 1/2$$

$$\frac{c_1}{2} + 3 - \frac{3}{2} c_1 = 1/2$$

$$-c_1 = 1/2 - 3 = 1/2 - 6/2 = -5/2$$

$$c_1 = 5/2;$$

$$c_2 = 2 - 5/2 = \frac{4}{2} - \frac{5}{2} = -1/2$$

$$y''(x) + b\,y'(x) + c\,y(x) = \emptyset$$

$$b^2 = 4c\,; \quad c = b^2/4$$

$$y''(x) + b\,y'(x) + \frac{b^2}{4}\,y(x) = \emptyset\,;$$

$$\frac{d^2}{dx^2}\,y(x) + b\,\frac{d}{dx}\,y(x) + \frac{b^2}{4}\,y(x) = \emptyset$$

$$\left(\frac{d}{dx} + \frac{b}{2}\right)^2 y(x) = \emptyset$$

$$\left(\frac{d^2}{dx^2} + b\,\frac{d}{dx} + \frac{b^2}{4}\right) y(x) = \emptyset$$

1c)

$$\cancel{y(x)\, e^{6x/2} = \frac{z_0\, e^{-6x/2}}{-6/2} + C}$$

$$\cancel{y(x) = C_1}$$

$$y(x)\, e^{6x/2} = z_0 x + C$$

$$\boxed{y(x) = z_0 x e^{-6x/2} + C e^{-6x/2}}$$

$$\left(\frac{d}{dx} + \frac{b}{2}\right)\underbrace{\boxed{\left(\frac{d}{dx} + \frac{b}{2}\right) y(x)}}_{z(x)} = 0$$

$$z'(x) + b/2\, z(x) = 0$$

$$z(x) = z_0 e^{-bx/2}$$

$$y'(x) + b/2\, y(x) = z_0 e^{-bx/2}$$

$$\left(y(x) e^{bx/2}\right)' = z_0 \cancel{e^{-bx/2}}$$

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