Newton's law of COOLING

T_a - ambient Temperature

$T(t)$ - temperature of an object

$$\frac{dT(t)}{dt} = -k(T(t) - T_A);$$

$$T(t=\emptyset) = T_0$$

$$\frac{dT(t)}{dt} + kT(t) = kT_A$$

$$\frac{d}{dt}\left(T(t)e^{kt}\right) = kT_A e^{kt}$$

$$T(t)e^{kt} = \frac{k T_A e^{kt}}{k} + C$$

$$T(t) = T_A + C^{-kt};$$

$$T(t=\emptyset) = T_0 = T_A + C;$$

$$C = T_0 - T_A$$

$$\boxed{T(t) = T_A + (T_0 - T_A)e^{-kt}}$$

$t = \emptyset$, $T(t=\emptyset) = T_0$

large t : $T(t) = T_A$

$$T(t) = T_A + (T_0 - T_A)e^{-kt};$$

<u>Example</u>

$t = \emptyset,\ T = 100°C$

$t = 10\,\text{minutes}:\ T(t) = 70°C$

$T_A = 20°C$, <u>when $T(t) = 50°C$</u>

<u>Sctn</u> $k = ?$ 10 minutes 70°C:

$$70°C = T(t = 10\,\text{minutes}) = 20°C + (100°C - 20°C) * e^{-k \cdot 10\,\text{minutes}}$$

$$70°C = 20°C + (100°C - 20°C)e^{-k\,10\,\text{min}}$$

$$\ln\left(\frac{70°C - 20°C}{100°C - 20°C}\right) = -k \cdot 10\,\text{minutes}$$

$$k = -\frac{1}{10\text{ minutes}} \ln\left(\frac{50}{80}\right)$$

$$= \frac{1}{10\text{ minutes}} \ln(8/5)$$

$$T(t) = T_A + (T_0 - T_A)e^{-kt} = 50\,^\circ C$$

$$e^{-kt} = \frac{50^\circ C - T_A}{T_0 - T_A}$$

$$-kt = \ln\left(\frac{50^\circ C - 20^\circ C}{100^\circ C - 20^\circ C}\right)$$

$$t = -\frac{1}{k}\ln(3/8) = 10 \text{ minutes} \; \frac{\ln(8/3)}{\ln(8/5)}$$

$$= 20.8 \text{ minutes}$$

$$\ln x^a = a \ln x$$

$$\ln(1/x) = -\ln(x)$$

Problem:

you enter a room, and kettle is at 70°C.

Wait 10 minutes, kettle is 50°C

IF $T_A = 20°C$, when the kettle was boiling?

Solution

$k = ?$ $\quad T(t) = T_A + (T_0 - T_A)e^{-kt}$

$$T(10\text{ minutes}) = 20°C + (100°C - 20°C)e^{-k \cdot 10\text{min}} = 50°C$$

$$-\left(\ln \frac{0°C - 20°C}{100°C - 20°C}\right) \cdot \frac{1}{10\text{ minutes}} = k$$

$$k = \frac{1}{10\text{ min}} \ln(8/5)$$

$$100°C = T(t) = 20°C + (70°C - 20°C)e^{-kt}$$

$$-\frac{1}{k} \ln \frac{100°C - 20°C}{70°C - 20°C} = t =$$

$$= -\frac{10 \text{ minutes}}{\ln(8/5)} \ln\left(8/5\right)$$

$t = \emptyset, \ 70°C$

$t = 10 \text{ minutes } 50°C$

$T_A = 20°C$

$t = X, \ 100°C; \ \cancel{X} = ?$

$$50°C = 20°C + (70°C - 20°C)\, e^{-k \cdot 10 \text{ minutes}}$$

$$\ln\left(\frac{5-2}{7-2}\right) = -k \cdot 10 \text{ minutes}$$

$$k = \frac{1}{10 \text{ minutes}} \ln(5/3)$$

$$100°C = 20°C + (70°C - 20°C)e^{-kx}$$

$$\ln \frac{100°C - 20°C}{70°C - 20°C} = -kx$$

$$x = -\frac{10 \text{ minutes}}{\ln(5/3)} \ln 8/5$$

$$= -10 \text{ minutes} \frac{\ln(8/5)}{\ln(5/3)} = -9.2 \text{ minutes}$$

Non linear cooling:

k is not a constant,

$$k \rightarrow k(T - T_a)^{1/4}$$

$$\frac{dT(t)}{dt} = -k(T - T_a)^{5/4}$$

$$\int \frac{dT}{(T-T_a)^{5/4}} = \int -k\,dt = -k\,t + C$$

$$-4(T - T_a)^{-1/4} = -k\,t + C$$

$$4\,(T-T_a)^{-1/4} = kt - c$$

$$4 = (kt-c)(T-T_a)^{1/4}$$

$$T - T_a = \left(\frac{4}{kt-c}\right)^4$$

$$T = T_a + \left(\frac{4}{kt-c}\right)^4$$

$$T(t=0) = T_a + \left(\frac{4}{+c}\right)^4 = T_o$$

$$\cancel{\frac{1}{c} = \frac{(T_o-T_a)^{1/4}}{256} \;;\; c = \frac{256}{}}$$

$$\frac{256}{C^4} = T_0 - T_a$$

$$\left(\frac{256}{T_0 - T_a}\right)^{1/4} = C$$

$$C = \frac{4}{\sqrt[4]{T_0 - T_a}} = \frac{4}{(T_0 - T_a)^{1/4}}$$

Second Newton law

$$F = m \cdot a = m \cdot \frac{d^2 x(t)}{dt^2}$$

2 steps: $F = m \cdot \frac{dv(t)}{dt} \Rightarrow$ solve $v(t)$

$$v(t) = \frac{dx(t)}{dt} \Rightarrow \text{solve } x(t)$$

Vertical motion

Drag: $F_d = - C\, v$ drag coef

Ball is dropped from 1000 meters from rest. $m = 2$ kg; $c = \frac{1}{2} \frac{kg}{sec}$

Questions:

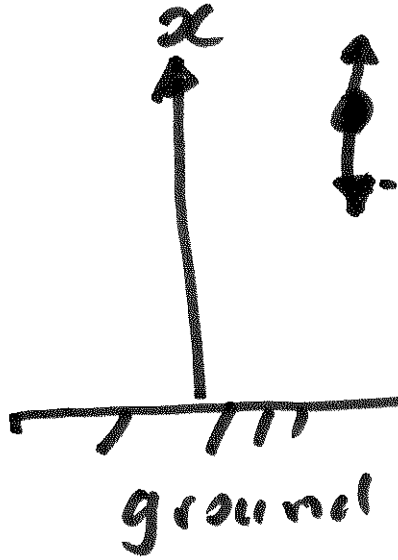
① Find Initial Value Problem for $v(t)$, $x(t)$;

② Solve these IVP (initial value problem)

③ Find terminal velocity

Terminal velocity is a velocity when gravity = drag

④ find T such that $x(T) = 0$



$F_{drag} = -cV$

$-mg = F_{gravity}$

$x(t=\emptyset) = 1000$ meters

$$F = -mg - cv = ma = m\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -g - \frac{c}{m}v(t)$$

$$\begin{cases} \dfrac{dv(t)}{dt} + \dfrac{c}{m}v(t) = -g \\ v(t=\emptyset) = \emptyset \end{cases}$$

Initial value problem for $v(t)$

$$\frac{dx(t)}{dt} = v(t);\quad x(t=\emptyset) = \emptyset$$

Question 2

$$\frac{dv(t)}{dt} + \frac{c}{m} v(t) = -g \; ; \; \mu(t) = e^{ct/m}$$

$$\frac{d}{dt}\left(v e^{ct/m}\right) = -g e^{ct/m}$$

$$v(t) e^{ct/m} = -\frac{gm}{c} e^{ct/m} + A$$

$$v(t) = -\frac{gm}{c} + A e^{-ct/m}$$

$$0 = v(t=0) = -\frac{gm}{c} + A \Rightarrow A = \frac{gm}{c}$$

$$v(t) = -\frac{gm}{c} + \frac{gm}{c} e^{-ct/m} = \frac{gm}{c}\left(-1 + e^{-ct/m}\right)$$

$$v(t) = \frac{gm}{c}\left(-1 + e^{-ct/m}\right)$$

$$c = \frac{1}{2}\,\frac{kg}{sec}$$

$$v(t) = \frac{10\text{ meters}\;\; 2\,kg}{sec^2 \;\; 1/2\;kg/sec} \cdot \left(-1 + e^{-t\cdot\frac{1}{4}\frac{kg}{sec\;kg}}\right)$$

$$v(t) = 40\,\frac{\text{meters}}{sec}\left(-1 + e^{-\frac{t}{4sec}}\right)$$