Modeling: $P(t)$ - population A

In + Out -

$$\frac{d}{dt} P(t) = r\,P(t);$$

r - rate of growth

$$P(t) = P_0 e^{rt};$$

$$P_0 \equiv P(t = \emptyset)$$

Town, r,
Time to double population

$$P(T) = P_0 e^{rT} = 2P_0$$

$$rT = \ln 2;$$

$$T = \frac{\ln 2}{r}$$

Town - it takes 5 years to double population. what is r?

$$r = \frac{\ln 2}{5 \text{ years}}$$

Population tripples every 2 years

T tripples

$$rT = \ln 3$$

$$r = \frac{\ln 3}{T} = \frac{\ln 3}{2\text{ years}}$$

C

$$P(t) = -\frac{A}{r} + Ce^{rt}$$

$$P(t=\emptyset) = P_0 = C - \frac{A}{r} \Rightarrow C = P_0 + \frac{A}{r}$$

$$P(t) = -\frac{A}{r} + \left(P_0 + \frac{A}{r}\right)e^{rt} =$$

$$= \frac{A}{r}\left(e^{rt} - 1\right) + P_0 e^{rt}$$

R

$$\frac{d}{dt} P(t) = r\, P(t) + A$$

normal growth

additional influx

$$\frac{d}{dt} P(t) - r\, P(t) = A; \quad e^{-rt}$$

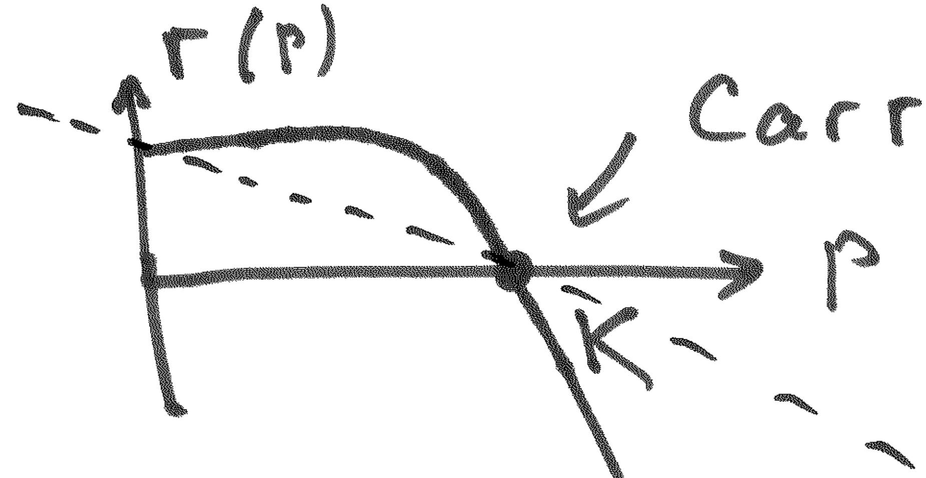
$$e^{-rt} \frac{d}{dt} P(t) - e^{-rt}\, r\, P(t)$$

$$\frac{d}{dt}\left(e^{-rt} P(t)\right) = A e^{-rt} = A e^{-rt};$$

$$e^{-rt} P(t) = \int A e^{-rt}\, dt = A\left(\frac{e^{-rt}}{-r}\right) + C$$

E

$$\frac{d}{dt} p(t) = r(p)\, p(t);$$



Carrying capacity

$$r(p) = r\left(1 - \left(\frac{K}{p}\right)^{-1}\right) = r\left(1 - \left(p/K\right)^{-1}\right)$$

Logistic equation

$$\cancel{\frac{d}{dt} p(t) = r\, p(t)\left(1 - \frac{K}{p(t)}\right)}$$

$$\frac{d}{dt} p(t) = \cdot$$

$$\frac{d}{dt} P(t) = r P(t)\left(1 - P(t)/K\right)$$

F

$$\int \frac{dP}{P(1 - P/K)} = \int r\, dt = rt + C$$

$$\int \frac{dP}{P(1 - P/K)} = \int \frac{K\, dP}{P(K-P)}$$

$$\frac{K}{(K-P)P} = \frac{A}{P} + \frac{B}{K-P} = \frac{1}{P} + \frac{1}{K-P}$$

$$= \frac{A(K-P) + BP}{P(K-P)} = \frac{P(B-A) + AK}{P(K-P)}$$

$A = 1 = B$

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$$\int \frac{dP}{P(1-P/K)} = \int \frac{dP}{P} + \int \frac{dP}{K-P}$$

$$= \ln P - \ln (K-P) =$$

$$= \ln \left(\frac{P}{K-P} \right) = rt + C$$

$$\frac{P(t)}{K-P(t)} = e^{rt+C} \qquad \emptyset < P < K$$

$$P(t) = e^{rt+C} K - P(t) e^{rt+C}$$

$$P(t) = \frac{e^{rt+C} K}{1 + e^{rt+C}}$$

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$$P(t) = \frac{K}{1 + e^{-rt-c}}$$

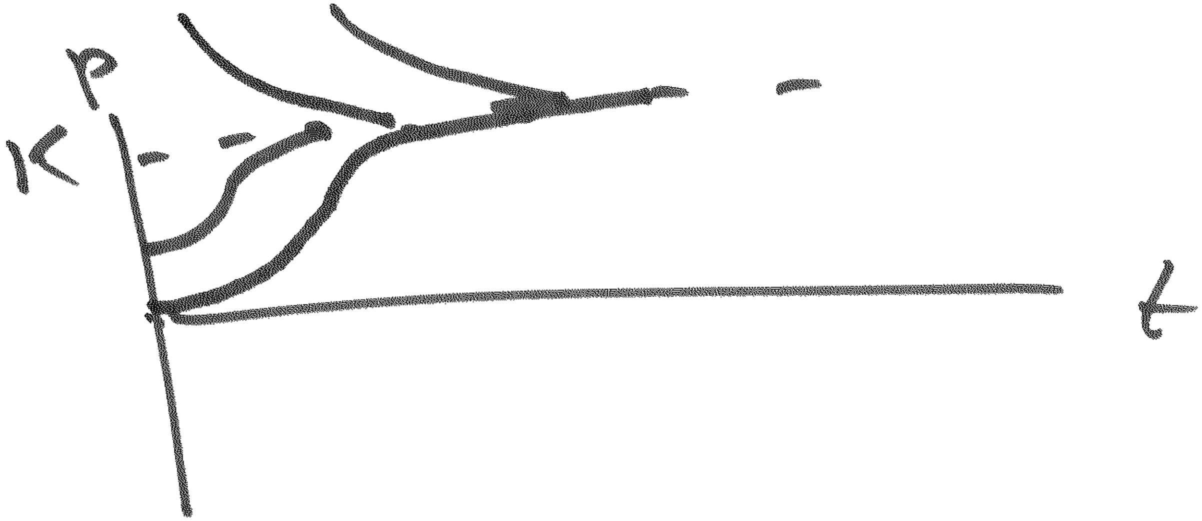
$$= \frac{K}{1 + \tilde{c}\, e^{-rt}} = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right) e^{-rt}}$$

$$P(t=0) = \frac{K}{1+\tilde{c}} = P_0 \qquad = \boxed{\frac{K P_0}{P_0 + (K - P_0) e^{-rt}} = P(t)}$$

$$K = P_0(1+\tilde{c}) = P_0 + P_0 \tilde{c}$$

$$\tilde{c} = \frac{K - P_0}{P_0} = \frac{K}{P_0} - 1$$

I



Radioactive decay

$$\frac{d}{dt} P(t) = -r\, P(t), \quad r > 0$$

$$P(t) = P_0 e^{-rt};$$

how fast does it take to decay to half of initial mass?

$$P_0 e^{-rT} = P_0/2$$

Half life time

$$e^{-rT} = 1/2$$

$$-rT = \ln 1/2 = -\ln 2$$

$$T = \frac{\ln 2}{r}$$

Mixing

K

φ liters per second enters a pool
concentration is c

φ liters leaves the pool

Volume is v

$Q(t)$ - amount of salt in a L pool.

$$\frac{d}{dt} Q(t) = \text{In} - \text{Out}$$

$$= C\,\varphi - \frac{Q(t)}{v}\varphi$$

liters per second entering

concentration of salt

$$\frac{d}{dt} Q(t) + \frac{Q(t)}{v}\varphi = C\varphi$$

$$\mu(t) = e^{\varphi t / v}$$

M

$$\frac{d}{dt}\left(Q(t)e^{\varphi t/\mathcal{V}}\right) = C\varphi e^{\varphi t/\mathcal{V}}$$

$$Q(t)e^{\varphi t/\mathcal{V}} = C\varphi \frac{e^{\varphi t/\mathcal{V}}}{\varphi/\mathcal{V}} + A$$

arbitrary constant

$$Q(t) = C\mathcal{V} + Ae^{-\varphi t/\mathcal{V}}$$

$$Q(t=0) = Q_0 = C\mathcal{V} + A;$$

$$A = Q_0 - C\mathcal{V}$$

$$Q(t) = C\mathcal{V} + (Q_0 - C\mathcal{V})e^{-\varphi t/\mathcal{V}}$$

$$= C\mathcal{V}\left(1 - e^{-\varphi t/\mathcal{V}}\right) + Q_0 e^{-\varphi t/\mathcal{V}}$$


Q(t)
cγ
t

$$\frac{d}{dt} Q(t) + Q(t)\frac{\varphi}{v} = c\varphi$$

0

$$y'(x) + p(x)y(x) = g(x);$$

$$\mu(x) = e^{\int p(x)\,dx}$$

$$Q \rightsquigarrow y$$

$$t \rightsquigarrow x$$

$$\varphi/v \rightsquigarrow p(x)$$

$$\mu(t) = e^{\int \frac{\varphi}{v} dt} = e^{\varphi/v \int dt} = e^{\varphi t/v}$$

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$$\frac{d}{dt} Q(t) + Q(t) \frac{\varphi(t)}{v(t)} = C(t)\, \varphi(t)$$

$$\mu(t) = e^{\int \frac{\varphi(t)}{v(t)} dt}$$

Newton's law of cooling Q

Given an object with initial temperature T what is the temperature of object as a function of time?

Rate of cooling is proportional to the difference between the temperature of an object and temperature of surroundings.

$$\frac{d}{dt} T(t) = k\big(T(t) - T_A\big)\cdot(-1)$$

$T_A \equiv$ ambient temperature.

$$\frac{d}{dt} T(t) = -\,k\,(T(t) - T_A)$$

$$\frac{d}{dt} T(t) + k\,T(t) = k\,T_A$$

$$\mu(t) = e^{\int dt\,(k)} = e^{kt}$$

$$\frac{d}{dt}\left(T(t)\,e^{kt}\right) = k\,T_A\,e^{kt}$$

T

$$T(t)e^{kt} = \frac{k T_A e^{kt}}{k} + C$$

$$T(t) = T_A + Ce^{-kt}$$

$$T(t=0) = T_0 = T_A + C$$

$$\Rightarrow C = T_0 - T_A$$

$$T(t) = T_A + (T_0 - T_A)e^{-kt}$$

$$= T_A\left(1 - e^{-kt}\right) + T_0 e^{-kt}$$

6)

$$T(t) = T_A\left(1 - e^{-kt}\right) + T_0 e^{kt}$$

$T_0 = 100\ °C$

$T(t = 20 \text{ minutes}) = 70\ °C$

what is rate of cooling