First order equation ordinary differential

- Separables Equations

$$y'(x) = F(y(x))\, G(x)$$

→ independent

→ dependent

$$\int \frac{dy}{F(y(x))} = \int G(x)\, dx$$

implicit

$y(x) = \ldots$ explicit

Integration factor method

$$y'(x) + p(x)\,y(x) = f(x) \quad \Big\| \times \underbrace{e^{\int p(x)dx}}_{=\mu(x)}$$

$$\frac{d}{dx}\left(y(x)e^{\int p(x)dx}\right) = f(x)e^{\int p(x)dx}$$

$$\Downarrow$$

$$y(x) = e^{-\int p(x)dx} \int f(x)e^{\int p(x)dx}\,dx$$

Modeling

$P(t)$

$\hookrightarrow$ population growth, radioactive decay, mixing, cooling/heating

mass

radioactive material

concentration

change in time of $P(t)$

$$\frac{d}{dt} P(t) = \underbrace{In(t)}_{\text{birth, immigration, move in}} - \underbrace{Out(t)}_{\text{death, emmigration, move out}}$$

Simple growth

$$\frac{d}{dt} p(t) = \alpha \, p(t);$$

$$p(t) = p(t=0)\, e^{\alpha t}$$

double, triple population

Logistic growth

simple

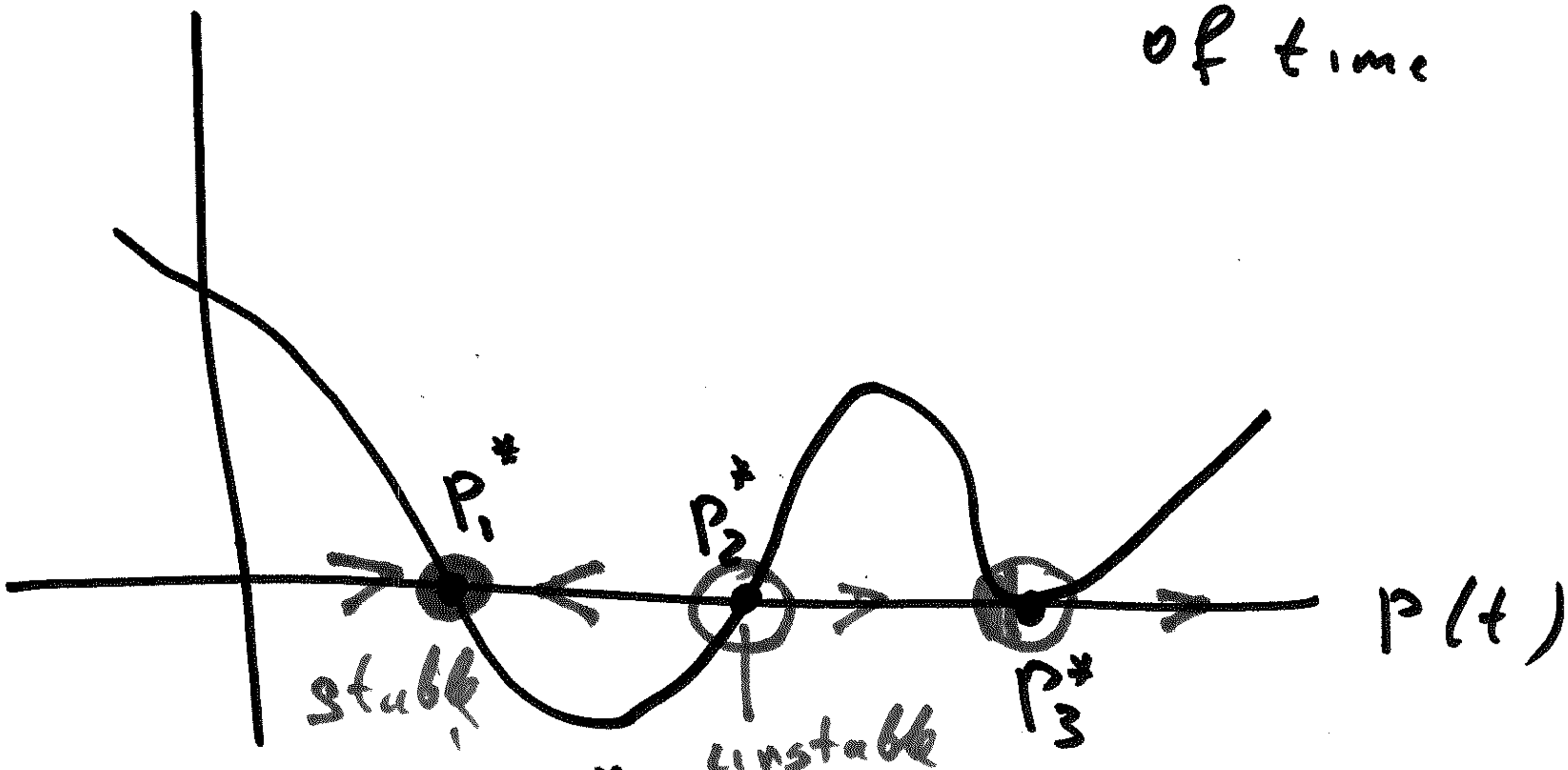
$$\frac{d}{dt} p(t) = \alpha \, p(t)\left(1 - \frac{p(t)}{N(t)}\right)$$

carrying capacity

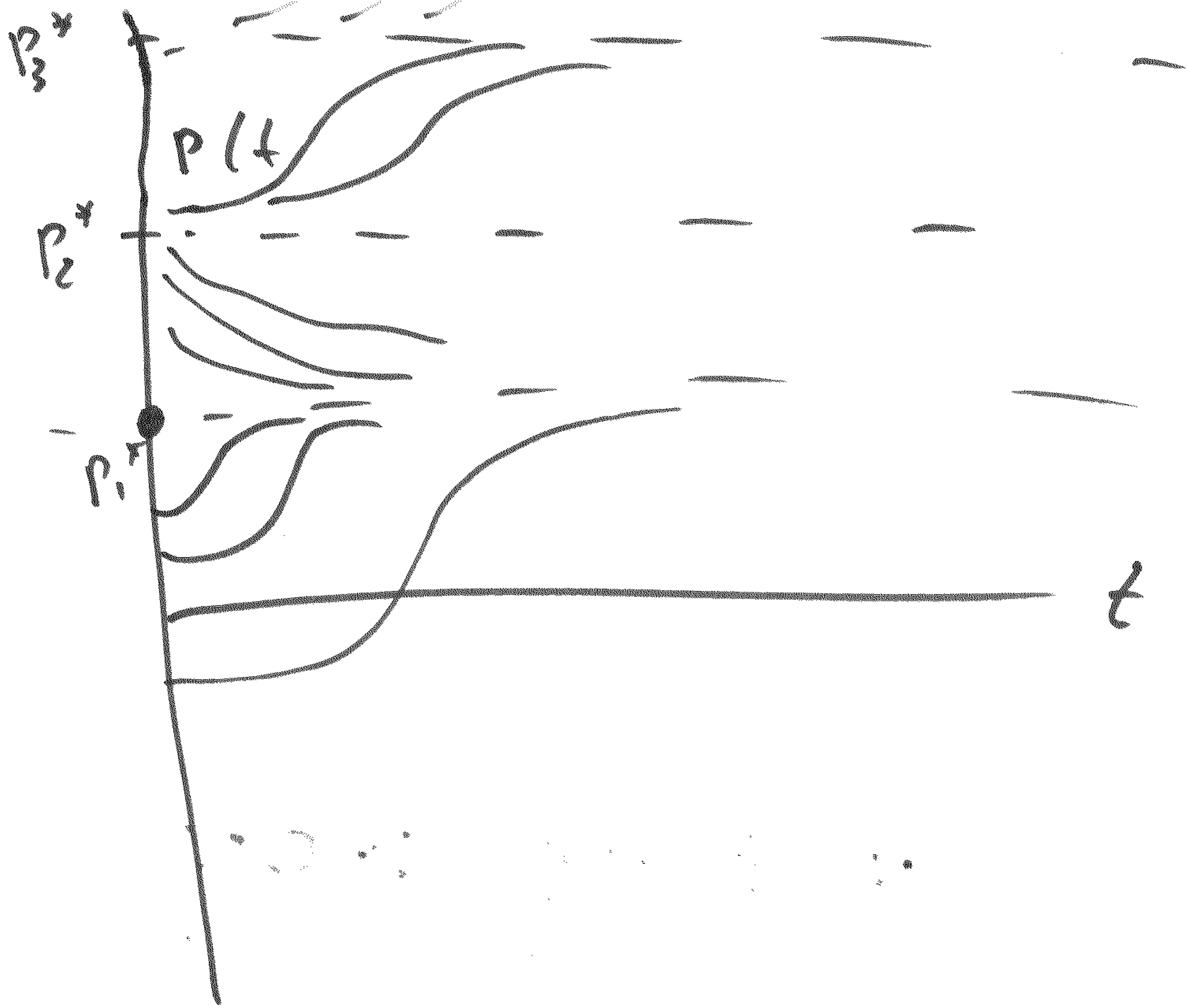
Phase plane

$$\frac{d}{dt} p(t) = f(p(t));$$

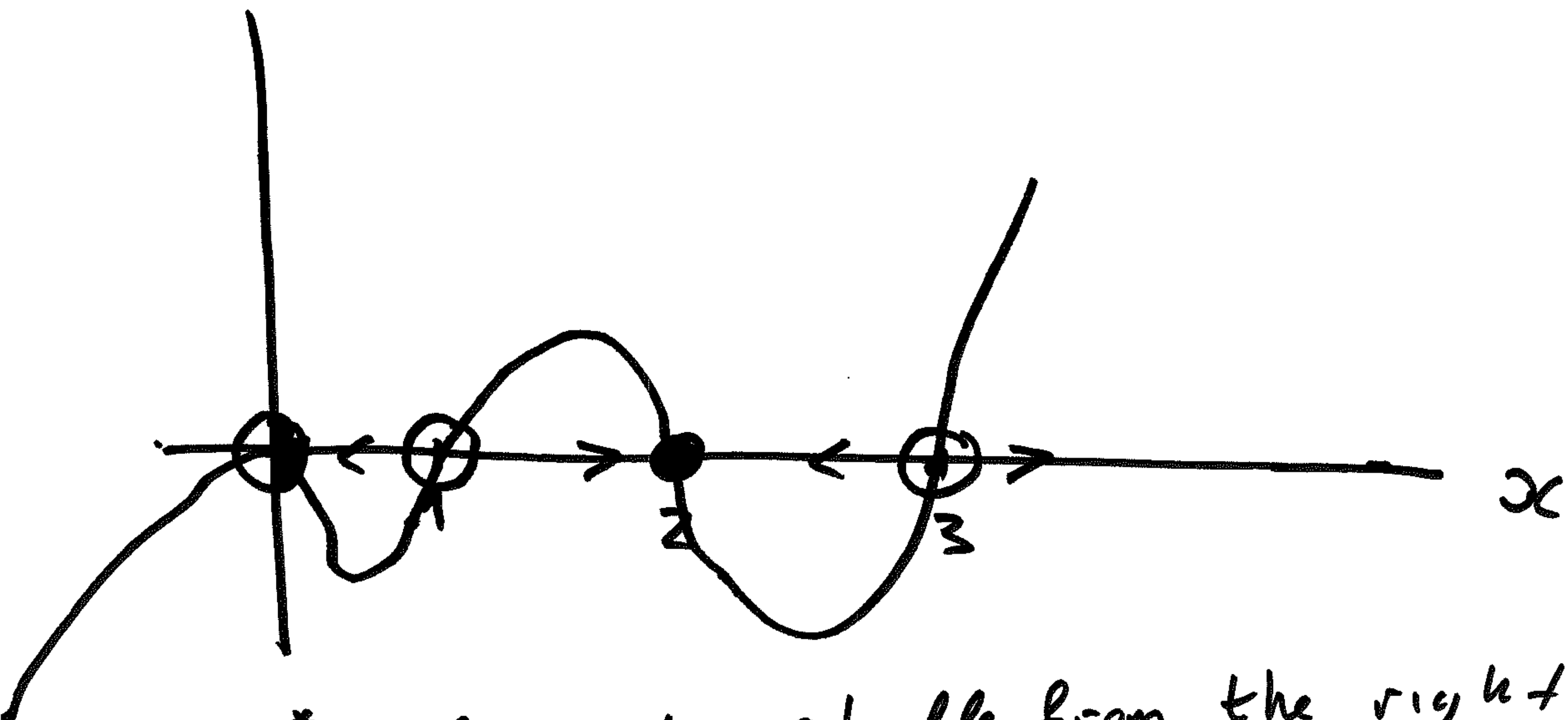
is not explicit function of time



Fixed points p^* are such that $f(p^*) = \emptyset$.


p_3^*
$p(t$
p_2^*
p_1^*
t

$$\frac{d}{dt} x(t) = x^2(t)\,(x(t)-1)\,(x(t)-2)\,(x(t)-3)$$



$x_1^* = 0$ mixed: stable from the right
unstable – " – left

$x_2^* = 1$ unstable

$x_3^* = 2$ stable

$x_4^* = 3$ unstable


3
2
1
t

Second order ordinary differential equations

$$\frac{d^2}{dt^2} x(t) = f\left(x(t), \frac{d}{dt}x(t), t\right) \quad :($$

Linear second order equations

$$y''(x) + p(x)y'(x) + g(x)y(x) = f(x)$$

homogeneous equation

$$y''(x) + p(x)y'(x) + g(x)y(x) = \emptyset. \qquad (*)$$

Find $y_1(x)$ and $y_2(x)$ that solve $(*)$

such that $W(y_1(x), y_2(x)) \equiv y_1(x)y_2'(x) - y_2(x)y_1'(x)$ is non zero

$y(x) = C_1 y_1(x) + C_2 y_2(x)$ is a general sltn.

Second order linear ODE with constant coefficients, homogeneous

$$a y''(x) + b y'(x) + c y(x) = 0 \qquad a, b, c \in \mathbb{R}$$

$$y(x) = e^{\lambda x}$$

$$a\lambda^2 + b\lambda + c = 0$$ - characteristic equation

then

1. $r_1 \neq r_2$ and both are real
2. $r_1 = r_2$ and is real
3. $r_1 = r_2^*$

$$a y''(x) + b y'(x) + c y(x) = f(x)$$

IF $f(x)$ is of "special" form
(polynomial, exponent, sin or cos)
then method of undetermined coefficients.

- IF $f(x) =$ sin or cos $\Rightarrow$ $y(x) = A \sin + B \cos$
- IF $f(x) =$ exp $\qquad y(x) =$ exp
- IF $f(x) =$ poly $\qquad y(x) =$ Polynomial

IF a particular solution
is a

No parts of a particular solution ~~sh~~ can be a part of a general solution of homogeneous equation.

$\times x$, $\times x^2$ to guarantee this

Method of variation of a parameter

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x) = f(x),$$

and $y_1(x), y_2(x)$ such that they solve

$$y_0''(x) + y_0'(x)\,p(x) + g(x)\,y_0(x) = \emptyset$$

then

$$y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

$$u_1(x) = -\int \frac{y_2(x) f(x)\,dx}{W(y_1(x), y_2(x))}$$

$$u_2(x) = \int \frac{y_1(x) f(x)\,dx}{W(y_1(x), y_2(x))}$$

$$\frac{d^2}{dt^2} x(t) + 2\gamma \frac{d}{dt} x(t) + \omega_0^2 x(t) = f \cos(\omega t)$$

- Free: $\gamma = f = 0$
- damped: $f = 0$, $\gamma > 0$
  - overdamped
  - critically damped
  - underdamped
- forced $\gamma = 0$, $f \neq 0$
  - nonresonant $\omega \neq \omega_0$
  - resonant $\omega = \omega_0$
- general formula

homogeneous $ax^2 y''(x) + bx\, y'(x) + c\, y(x) = 0$

$y(x) = x^\lambda$

quadratic equation for λ

$a\lambda(\lambda-1) + b\lambda + c = 0$

- $\lambda_1 \neq \lambda_2$, both real
- $\lambda_1 = \lambda_2$, real
- $\lambda_1 = \lambda_2^*$.

inhomogeneous Euler

$ax^2 y''(x) + bx\, y'(x) + c\, y(x) = f(x)$

Systems of linear equations

$$\frac{dx(t)}{dt} = a\,x(t) + b\,y(t)$$

$$\frac{dy(t)}{dt} = c\,x(t) + d\,y(t).$$

$$x(t = t_0) = x_0$$

$$y(t = t_0) = y_0$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Eigenvector $\underline{u}$, and eigenvalue λ

$$\underline{\underline{A}}\,\underline{u} = \lambda \underline{u} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A - \lambda I) = \emptyset$$

$$(a-\lambda)(d-\lambda) - bc = \emptyset$$

$$(\lambda - a)(\lambda - d) - bc = \emptyset$$

$$\lambda^2 - (a+d)\lambda + ad - bc$$

$$\boxed{\text{Solve } \underline{\underline{A}}\,\underline{u} = \lambda u}$$

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda_{1,2} = \frac{\text{Trace}(A) \pm \sqrt{Tr^2(A) - 4\,Det(A)}}{2}$$

- If $\lambda_1 \neq \lambda_2$ and both are real

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \underline{u}_1 e^{\lambda_1 t} + C_2 \underline{u}_2 e^{\lambda_2 t} \qquad (*)$$

u_1, u_2 are eigenvectors

- If $\lambda_1 = \lambda_2$

$\underline{u}_1 \neq \underline{u}_2$ $(*)$ is a solution

$\underline{u}_1 = \underline{u}_2$ – one eigenvector, matrix is deffective

Find $\underline{a}$ s. that

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{a} = \underline{u}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \underline{u} e^{\lambda t} + C_2 (\underline{u}\, t + \underline{a}) e^{\lambda t}$$

IF $\lambda_1 = \lambda_2^*$

then ....

$$\frac{d}{dt} x(t) = f(x(t), y(t))$$

$$\frac{d}{dt} y(t) = g(x(t), y(t))$$