$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f(x(t), y(t)) \\ g(x(t), y(t)) \end{pmatrix}$$

Fixed points (x^*, y^*)

$$f(x^*, y^*) = g(x^*, y^*) = 0,$$

$$\ddot{\theta}(t) + \sin\theta(t) = 0$$

$$\begin{cases} \dot{\theta}(t) = \omega(t) \\ \dot{\omega}(t) = -\sin\theta(t) \end{cases}$$

$\theta^* = 0,\ \omega^* = 0$

$\theta^* = \pi,\ \omega^* = 0$ $\Rightarrow$ $\theta^* = \pi n,\ n = 0, \pm 1, \pm 2$

$\omega^* = 0$

$$x(t) = x^* + \eta(t) \qquad \eta(t) \longrightarrow \text{eta}$$

$$y(t) = y^* + \nu(t)$$

$$\frac{d}{dt}\left(\underbrace{\cancel{x^*}}_{=0} + \eta(t)\right) = f\left(x^* + \eta(t), y^* + \nu(t)\right) \overset{A}{=}$$

$$\frac{d}{dt}\left(\underbrace{y^*}_{=0} + \nu(t)\right) = g\left(x^* + \eta(t), y^* + \nu(t)\right) =$$

$$= \underbrace{g(x^* y^*)}_{=0} + \frac{\partial g}{\partial x}(x^* y^*)\,\eta(t) + \frac{\partial g}{\partial y}(x^* y^*) \cdot \nu(t)$$

$$\overset{A}{=} \underbrace{f(x^*, y^*)}_{=0} + \frac{\partial f(x^* y^*)}{\partial x} \cdot \eta(t) + \frac{\partial f(x^* y^*)}{\partial y}\,\nu(t)$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta(t) \\ \nu(t) \end{pmatrix} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}\Bigg|_{(x^*, y^*)} \cdot \begin{pmatrix} \eta(t) \\ \nu(t) \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix};$$

Eigenvalues: $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$

$$\mathrm{Det}(A - \lambda I) = \emptyset,$$

$$\mathrm{Det}(A - \lambda I) = \mathrm{Det}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \mathrm{Det}\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (\lambda - a)(\lambda - d) - bc$$

$$= \lambda^2 - (a+d)\lambda + \underline{ad - bc} = \emptyset$$

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\boxed{\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4\,Det(A)}}{2}}$$

Solve $\underline{\underline{A}}\,\underline{v}_1 = \lambda_1 \underline{v}_1$ and $\underline{\underline{A}}\,\underline{v}_2 = \lambda_2 \underline{v}_2$

$$A = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$$

$Tr = 6$

$Det = 8$

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4ad}}{2}$$

$$= \frac{a+d \pm \sqrt{(a-d)^2}}{2}$$

$$= \frac{a+d \pm |a-d|}{2} = a, d$$

$\lambda_1 = 2$

$\lambda_2 = 4$

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{\underline{A}}\,\underline{v}_1 = \lambda_1 \underline{v}_1 ; \quad \underline{\underline{A}}\,\underline{v}_2 = \lambda_2 \underline{v}_2$$

$$\underline{x}(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t}$$ if λ_1, λ_2 are real and $\lambda_1 \neq \lambda_2$

2x2 matrix is deffective if it has only one eigen vector.

$$A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$$

- $$\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$2x_1 = 2x_1$

$5x_1 + 4x_2 = 2x_2$

$5x_1 = -2x_2$ $\qquad \underline{v}_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

- $$\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 4\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$2x_3 = 4x_3 \Rightarrow x_3 = \emptyset$ $\qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$5x_3 + 4x_4 = 4x_4 \Rightarrow$

Let λ_1 and λ_2 be complex,

$\lambda_1 = \lambda_2^*$, IF a, b, c, d are real

$\lambda_1 = \gamma + i\omega \quad \gamma = \mathrm{Re}\,\lambda_1 = \mathrm{Re}\,\lambda_2$

$\lambda_2 = \gamma - i\omega \quad \omega = \mathrm{Im}\,\lambda_1 = -\mathrm{Im}\,\lambda_2$

$\underline{v}_1 = \underline{p} + i\underline{q} \quad \underline{p} = \mathrm{Re}\,\underline{v}_1 = \mathrm{Re}\,\underline{v}_2$

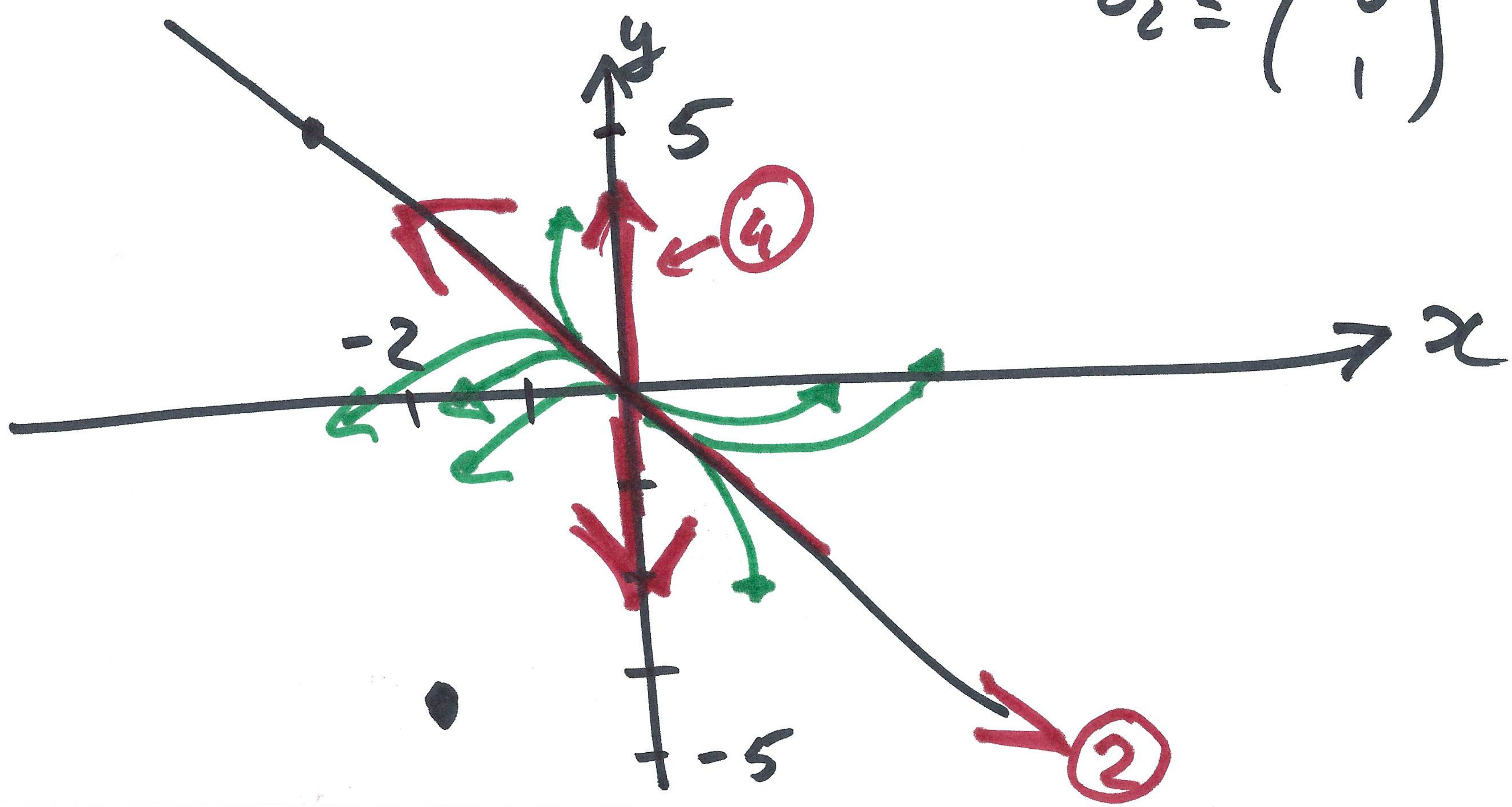
$\underline{v}_2 = \underline{p} - i\underline{q} \quad \underline{q} = \mathrm{Im}\,\underline{v}_1 = -\mathrm{Im}\,\underline{v}_2$

$$x(t) \overset{?}{=} C_1(\underline{p}\cos\omega t - \underline{q}\sin\omega t)e^{\gamma t} + c_2(\underline{p}\sin\omega t + \underline{q}\cos\omega t)e^{\gamma t}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$(\emptyset, \emptyset)$ is unstable fixed point

$\lambda_1 = 2;\ v_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ $\quad \lambda_2 = 4 \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$\lambda_{12} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4\,Det\,A}}{2}$$

$$A = \begin{pmatrix} -2 & -1 \\ 7 & 2 \end{pmatrix} \quad \begin{matrix} Tr\,A = \varnothing \\ Det = -4 + 7 = 3 \end{matrix}$$

$$\lambda_{1,2} = \frac{0 \pm \sqrt{-12}}{2} = \pm\sqrt{-3} = \pm i\sqrt{3} = \pm\sqrt{3}\,i$$

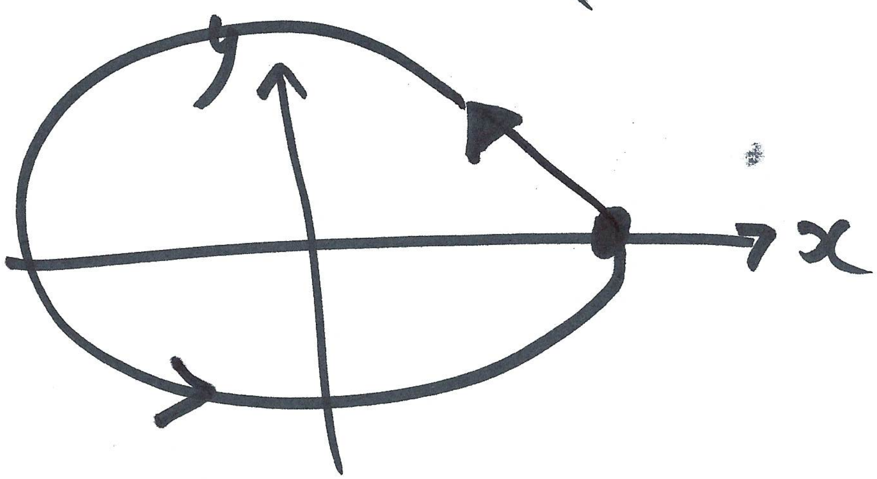
$$\frac{d}{d} \begin{pmatrix} -2 & -1 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i\sqrt{3}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad v = \begin{pmatrix} 1 \\ -2-i\sqrt{3} \end{pmatrix}$$

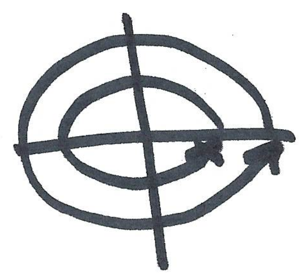
$$-2x_1 - x_2 = i\sqrt{3}\,x_1; \quad \boxed{(-2-i\sqrt{3})}\,\boxed{x_1} = \boxed{x_2} \quad 1$$

$$x(t) = C_1\left(\begin{pmatrix}1\\-2\end{pmatrix}\cos\sqrt{3}t - \begin{pmatrix}0\\-\sqrt{3}\end{pmatrix}\sin\sqrt{3}t\right)$$

$$+ C_2\left(\begin{pmatrix}1\\-2\end{pmatrix}\sin\sqrt{3}t + \begin{pmatrix}0\\-\sqrt{3}\end{pmatrix}\cos\sqrt{3}t\right)$$

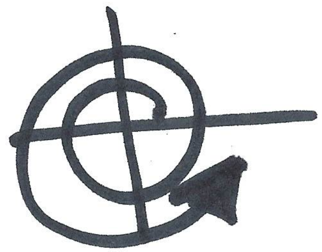
$$\begin{pmatrix}-2 & -1\\ 7 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2\\7\end{pmatrix}$$





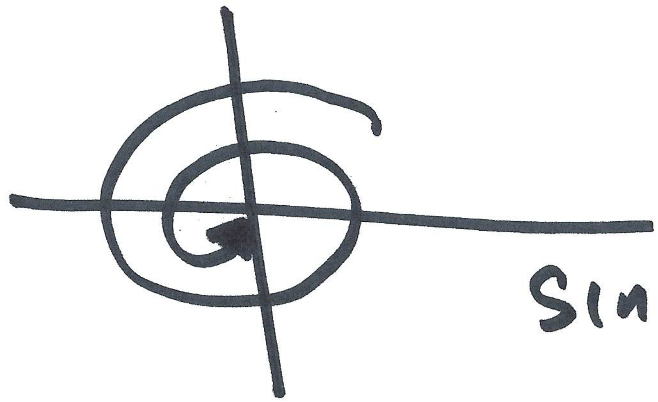
(∅,∅)
center

IF $\mathrm{Re}\,\lambda > 0$, $(0,0)$ is unstable



Source

IF $\mathrm{Re}\,\lambda < 0$



Sink

IF $\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = \emptyset, \quad \lambda_1 \neq \lambda_2$

$$\underline{x}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

IF $\cancel{\mathrm{Im}\,\lambda} \quad \mathrm{Im}\,\lambda_1 = -\mathrm{Im}\,\lambda_2 \neq \emptyset,$

$$x(t) = c_1 (\underline{p} \cos \omega t - \underline{q} \sin \omega t) e^{\gamma t} + c_2 (\underline{p} \sin \omega t + \underline{q} \sin \omega t) e^{\gamma t}$$

Deffective matrix

has only one eigenvalue and
only one eigen vector

$$\underline{x}(t) = c_1 \underline{v} e^{\lambda t} + c_2 (\underline{v} t + \underline{a}) e^{\lambda t}$$

$$(A - \lambda I)\underline{a} \stackrel{?}{=} \underline{v}$$

$$\boxed{(\underline{\underline{A}}\,\underline{a} - \lambda \underline{a}) = \underline{v}}$$

$$x(t) = (\underline{v} t + a) e^{\lambda t}$$

$$\dot{x}(t) = (\underline{v} + \lambda \underline{v} t + \lambda \underline{a}) e^{\lambda t}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(\underline{v} t + a) e^{\lambda t} = (\lambda \underline{v} t + \underline{\underline{A}}\, a) e^{\lambda t}$$

$$\underline{\dot{x}} - \underline{\underline{A}}\,\underline{x} = \underline{v} + \cancel{\lambda \underline{v}} t + \lambda \underline{a} - \cancel{\lambda \underline{v}} t - A a) e^{\lambda t} = \emptyset$$

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}; \quad \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

Solve $\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\lambda = 3;$

$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}; \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$3a + 2b = 3a \Rightarrow b = \emptyset$

$b = b$

$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solve $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{a} = v$

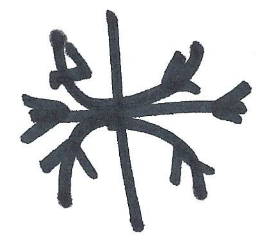
$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \underline{a} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$0 \cdot u + 2 \cdot v = 1; \quad v = 1/2; \quad u = 0$$

$$\underline{a} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \qquad x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} t \\ 1/2 \end{pmatrix} e^{3t}$$

- Stability of $(\emptyset,\emptyset)$
  - IF $\lambda_1 < \lambda_2 < \emptyset$, $(\emptyset,\emptyset)$ is stable



IF one of ~~the~~ λ's are larger then $\emptyset$, then $(\emptyset,\emptyset)$ is unstable

IF $\text{Re}\,\lambda_1 = \text{Re}\,\lambda_2 < \emptyset$ stable

IF $\text{Re}\,\lambda_1 = \text{Re}\,\lambda_2 > \emptyset$ unstable

IF $\text{Det}\,A < \emptyset$, ~~solution~~ $(\emptyset,\emptyset)$ is unstable

IF $\text{Tr}\,A > \emptyset$ -//-

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ -\sin\theta\end{pmatrix}$$

$\theta = \omega = \phi$

$\theta = \pi,\ \omega = \phi$

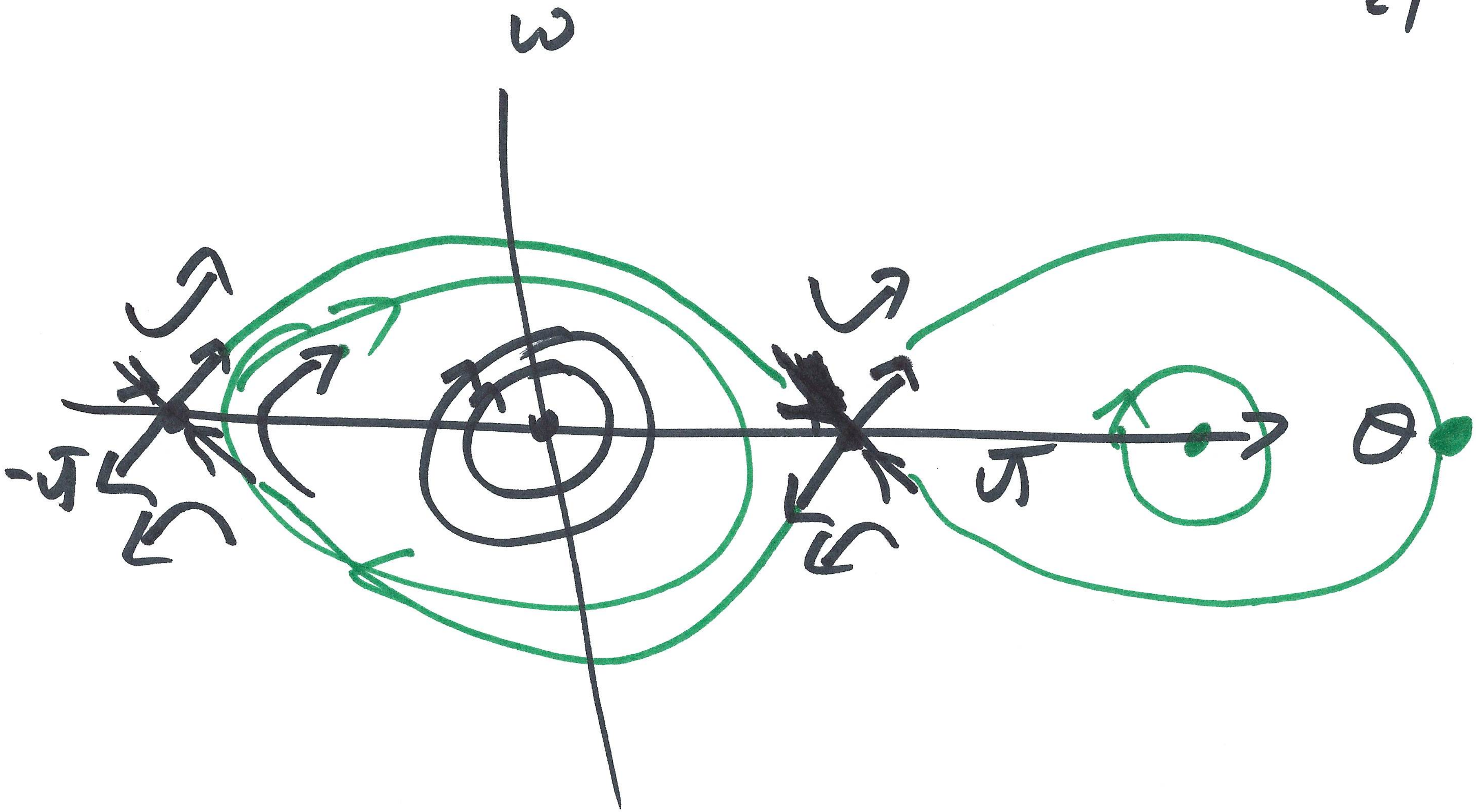
$$J = \begin{pmatrix}\phi & 1\\ -\cos\theta & \phi\end{pmatrix}$$

$$J(\phi,\phi) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$\lambda_{1,2} = \pm i$

$$J(\phi,\pi) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$v_1 = \begin{pmatrix}1\\1\end{pmatrix}$ $\lambda_1 = 1,$ $\lambda_2 = -1$ $v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$


ω
θ
π
-π