Nov 21

Initial value problem

$$\begin{cases} \frac{d}{dt} x(t) = f(x(t), y(t)), \\ \frac{d}{dt} y(t) = g(x(t), y(t)). \\ \text{Initial conditions} \quad x(t=t_0) = x_0 \\ \qquad\qquad\qquad\quad\; y(t=t_0) = y_0 \end{cases}$$

System of first order nonlinear ordinary differential equation

Fixed point
steady state
critical points

(x^*, y^*) $\quad f(x^*, y^*) = g(x^*, y^*) = 0$

$$x(t) = x^* + \eta(t), \quad |\eta(t)| \ll 1$$

$$y(t) = y^* + \nu(t). \quad |\nu(t)| \ll 1$$

$$\frac{d\,\eta(t)}{dt} = \left(\frac{\partial f}{\partial x}\,\eta(t) + \frac{\partial f}{\partial y}\,\nu(t)\right)\Bigg|_{(x^*,y^*)}$$

$$\frac{d\,\nu(t)}{dt} = \frac{\partial g}{\partial x}\,\eta(t) + \frac{\partial g}{\partial y}\,\nu(t)\Bigg|_{(x^*,y^*)}$$

$$\frac{d}{dt}\begin{pmatrix}\eta(t)\\ \nu(t)\end{pmatrix} = \begin{pmatrix}\partial f/\partial x & \partial f/\partial y\\ \partial g/\partial x & \partial g/\partial y\end{pmatrix}_{(x^* y^*)}\begin{pmatrix}\eta(t)\\ \nu(t)\end{pmatrix}.$$

$$\lambda_{1,2} = \frac{\mathrm{Tr}(J) \pm \sqrt{\mathrm{Tr}^2(J) - 4\,\mathrm{Det}(J)}}{2}$$

$$x(t) = x^* + \eta(t)$$

$$y(t) = y^* + \nu(t)$$

$$\lambda_{1,2} = \frac{Tr(J) \pm \sqrt{(Tr(J))^2 - 4\,Det(J)}}{2} \;; \quad \lambda_1 \curvearrowright \underline{v}_1, \quad \lambda_2 \curvearrowright \underline{v}_2 = \underline{v}_2$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix} + c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$$

- (x^*, y^*) is stable fixed point

  IF $Re\,\lambda_1 < 0$, $Re(\lambda_2) < 0$.

- IF $Re(\lambda_1) > 0$ or $Re(\lambda_2) > 0$ (or both)

  then (x^*, y^*) is unstable

- IF $Det(J) < \varnothing$ then (x^*, y^*) is unstable

$$\sqrt{Tr^2(J) - 4Det} > Tr(J)$$

$$Tr(J) + \sqrt{Tr^2(J) - 4Det(J)} > \varnothing$$

- IF $Tr(J) > \varnothing$

$$Tr(J) + \sqrt{Tr^2(J) - 4Det(J)} > \varnothing$$

$$\frac{d}{dt} R(t) = R(t)\left(3 - R(t) - 2S(t)\right)$$

$$\frac{d}{dt} S(t) = S(t)\left(2 - R(t) - S(t)\right)$$

$(\emptyset,\emptyset)$ $\lambda_1 = 2$ $\begin{pmatrix}1\\0\end{pmatrix}$

unstable $\lambda_2 = 3$ $\begin{pmatrix}0\\1\end{pmatrix}$

$(3,\emptyset)$ $\lambda_1 = -3$ $\begin{pmatrix}1\\0\end{pmatrix}$

stable $\lambda_2 = -1$ $\begin{pmatrix}3\\-1\end{pmatrix}$

$(\emptyset,2)$ $\lambda_1 = -1$ $\begin{pmatrix}1\\-2\end{pmatrix}$

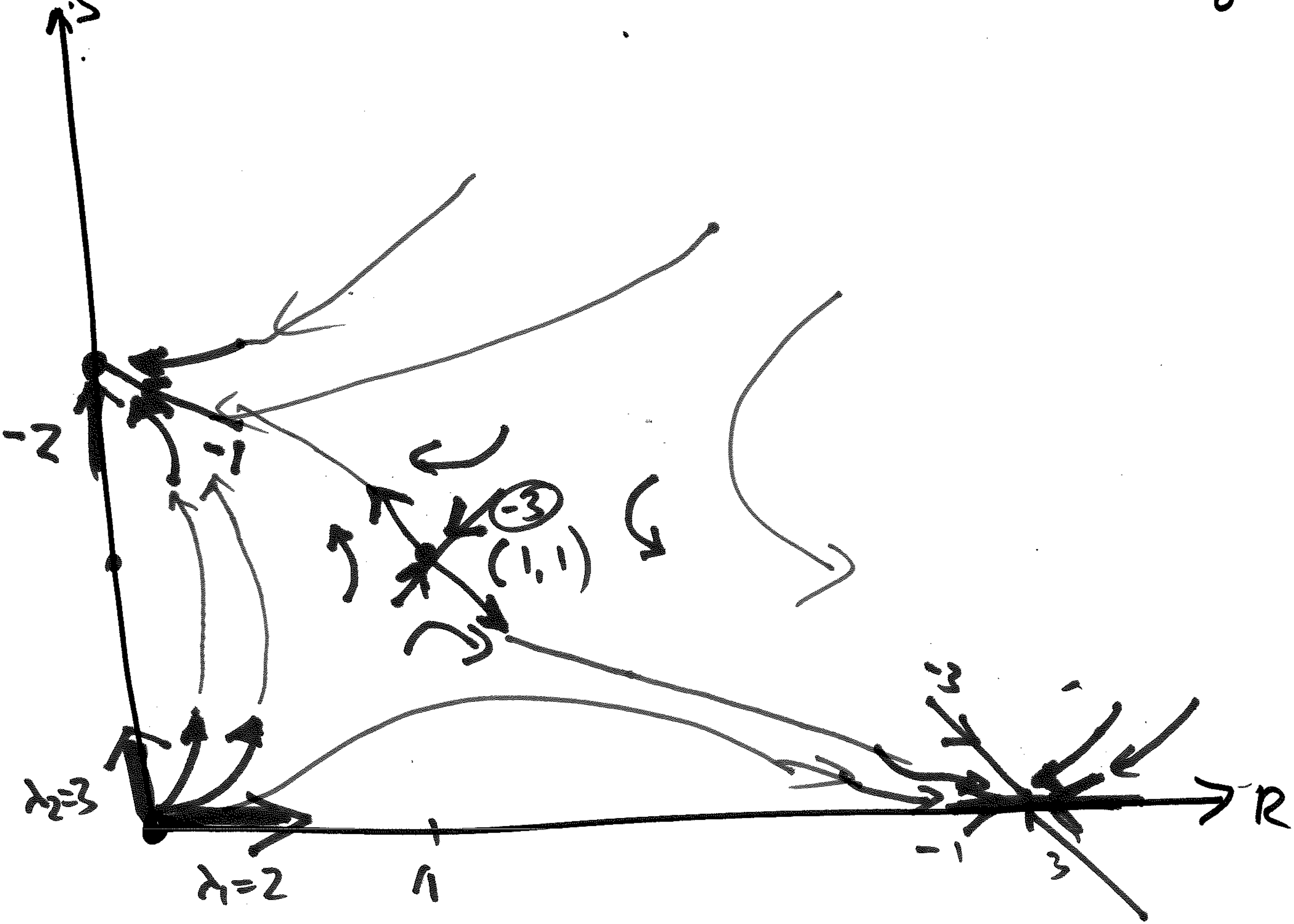
$\lambda_2 = -2$ $\begin{pmatrix}0\\1\end{pmatrix}$

stable

$(1,1)$ $\lambda_1 = 1$; $\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}$

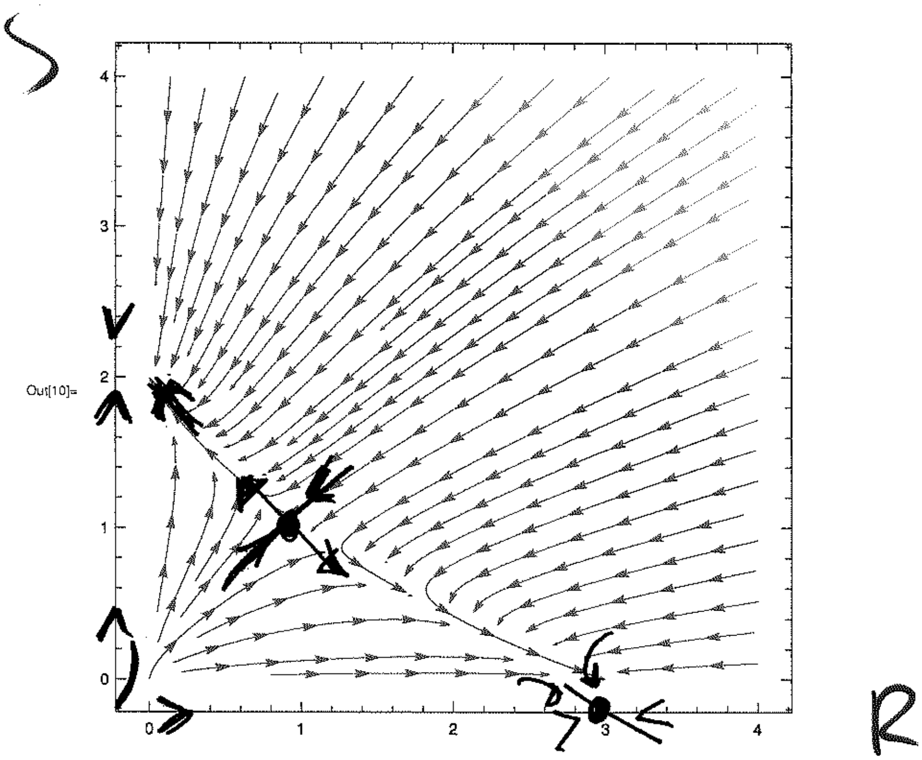
$\lambda_2 = -3$ $\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}$

unstable


S
R
-2
-1
-3
(1,1)
-3
-1
3
$\lambda_2=3$
$\lambda_1=2$
1

In[10]:= StreamPlot[{x (3 - x - 2 y), y (2 - x - y)}, {x, 0, 4}, {y, 0, 4}]

Out[10]=



In[8]:= Eigenvectors[{{-1, 0}, {-2, -2}}]

Out[8]= {{0, 1}, {-1, 2}}

Determine Steady States, their stability & and plot plausable phase diagram.

$$\begin{cases} \dot{x} = 1 - 2x - y - xy = f(x,y) \\ \dot{y} = 3xy - y \end{cases}$$

Solution

$$\dot{y} = g(x,y) = y(3x-1) \nearrow y = \emptyset,\ 1 - 2x = \emptyset \Rightarrow x = \tfrac{1}{2} : (\tfrac{1}{2}, \emptyset)$$

$$\searrow x = \tfrac{1}{3} : 1 - \tfrac{2}{3} - y(\tfrac{1}{3} + 1) = \emptyset$$

$$\tfrac{1}{3} - \tfrac{4}{3}y = 0$$

$$1 - 4y = y \qquad (\tfrac{1}{3}, \tfrac{1}{4})$$

$$y = \tfrac{1}{4}$$

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} =$$

$$= \begin{pmatrix} -2-y & -1-x \\ 3y & 3x-1 \end{pmatrix} = J(x,y)$$

$$J(x,y) = \begin{pmatrix} -2-y & -1-x \\ 3y & 3x-1 \end{pmatrix}$$

$(x^*, y^*) = (1/2, \varnothing)$ - unstable

$$J = \begin{pmatrix} -2 & -3/2 \\ 0 & 1/2 \end{pmatrix}$$

$\lambda_1 = -2;\ v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$-2a - 3/2\, b = -2a$

$-3/2\, b = \varnothing \Rightarrow b = \varnothing$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1/2;\ v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$

$-2c - \frac{3}{2} d = \frac{1}{2} c$

$4c + 3d = -c$

$5c = -3d$

$v_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$$(x^*, y^*) = (1/3, 1/4)$$

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$$J(1/3, 1/4) = \begin{pmatrix} -9/4 & -4/3 \\ 3/4 & 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \frac{-9-\sqrt{17}}{6} \\ 1 \end{pmatrix}$$

$$Tr = -9/4$$

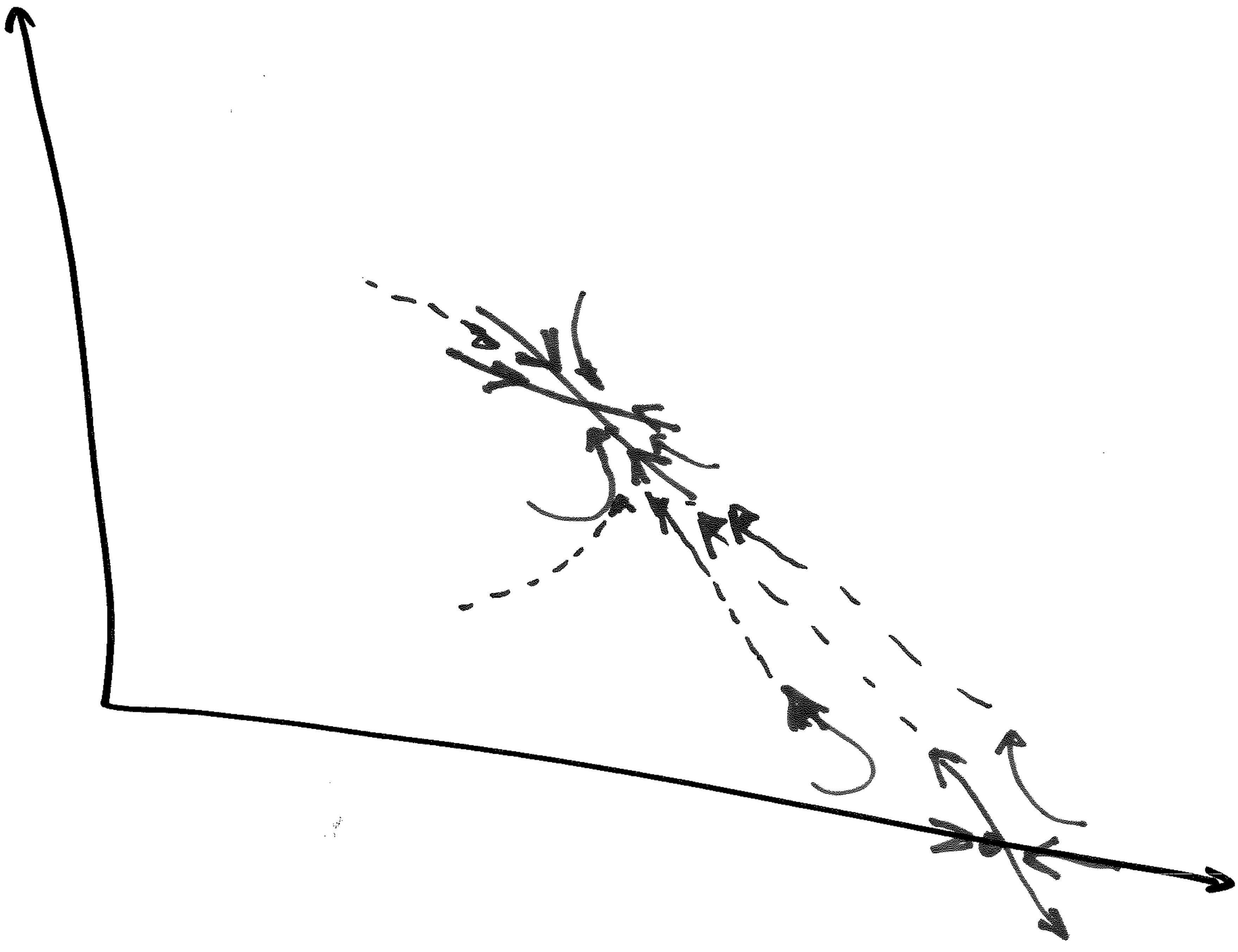
$$Det = \frac{4 \cdot 3}{3 \cdot 4} = 1$$

$$\lambda_{1,2} = \frac{-9/4 \pm \sqrt{\frac{81}{16} - 4}}{2} =$$

$$= -\frac{9}{8} \pm \frac{\sqrt{17}}{2 \cdot 4}$$

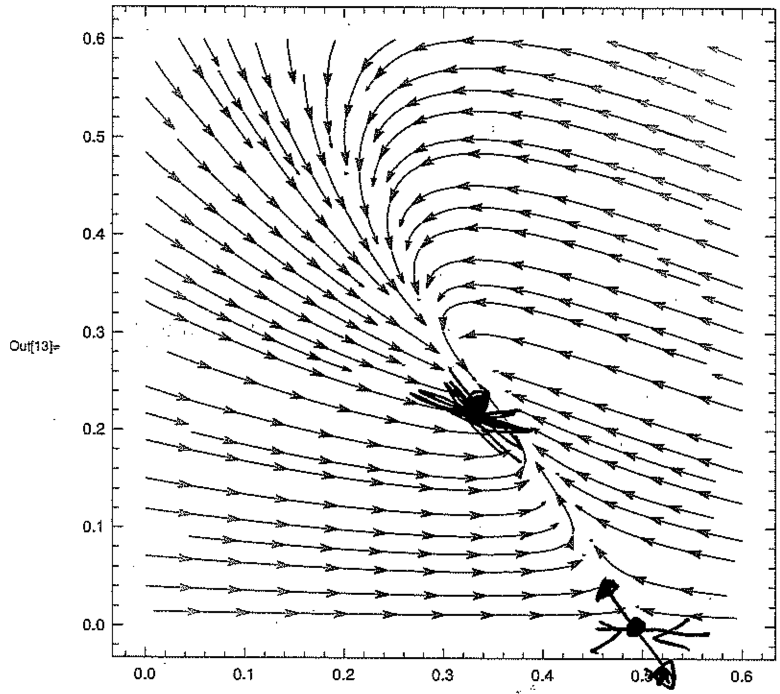
$$= -\frac{9}{8} \pm \frac{\sqrt{17}}{8} < 0$$

$$v_1 = \begin{pmatrix} \frac{-9+\sqrt{17}}{6} \\ 1 \end{pmatrix} \qquad \boxed{\text{stable}}$$



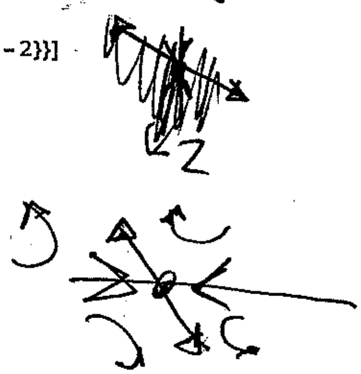
In[13]:= StreamPlot[{1 - 2 x - y - x y, 3 x y - y}, {x, 0, 0.6}, {y, 0, 0.6}]

Out[13]=



In[8]:= Eigenvectors[{{-1, 0}, {-2, -2}}]

Out[8]= {{0, 1}, {-1, 2}}



$\dot{x} = f(x,y)$

$\dot{y} = g(x,y)$

$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

periodic solution

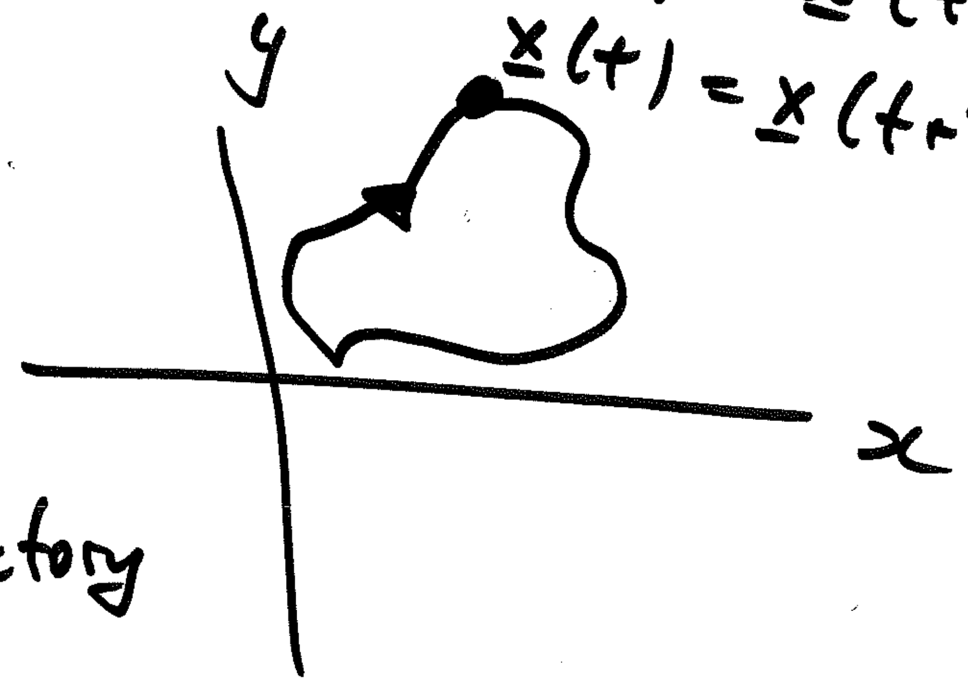
there exists $T > 0$,

such that $\underline{x}(t) = \underline{x}(t+T)$

$\underline{x}(t) = \underline{x}(t+T)$

$x(t+T) = x(t)$

$y(t+T) = y(t)$



periodic solution

$\equiv$

closed trajectory

mass-spring

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{k/m}$$

$$x(t) = R\cos(\omega t + \varphi)$$

$$\dot{x}(t) = -R\omega\sin(\omega t + \varphi)$$

$$\left(\frac{x(t)}{R}\right)^2 + \left(\frac{\dot{x}(t)}{R\omega}\right)^2 = 1$$