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Nov 20

$$\left.\begin{aligned}\frac{d}{dt}x(t) &= f(x(t), y(t)) \\ \frac{d}{dt}y(t) &= g(x(t), y(t))\end{aligned}\right\}$$

Example: $\ddot{\theta} + \sin\theta = \emptyset$

$$\Rightarrow \omega = \frac{d\theta}{dt}$$

$$\frac{d}{dt}\omega = -\sin\theta$$

$$\frac{d}{dt}\begin{pmatrix}\theta(t) \\ \omega(t)\end{pmatrix} = \begin{pmatrix}\omega(t) \\ -\sin\theta(t)\end{pmatrix}$$

$$x^2 y''(x) + x y'(x) + y(x) = 0$$

$$\cancel{+u(x)}\quad y'' = -\frac{y'(x)}{x} - y(x)/x^2$$

$$y'(x) = u(x)$$

$$\frac{d}{dx}\begin{pmatrix} y(x) \\ u(x) \end{pmatrix} = \left.\begin{matrix} u(x) \\ -\frac{y'(x)}{x} - y(x)/x^2 \end{matrix}\right\}$$

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Lotka-Voltera system

$R(t)$ - rabbits

$S(t)$ - sheeps

competition

$$\frac{d}{dt}R(t) = R(t)\left(3 - R(t) - 2S(t)\right)$$

$$\frac{d}{dt}S(t) = S(t)\left(2 - S(t) - R(t)\right)$$

simple reproductions

Logistic equation

Lotka - Voltera

Steady state

$$\frac{d}{dt} x(t) = f(x(t), y(t))$$

$$\frac{d}{dt} g(t) = g(x(t), y(t))$$

IF $x = x^*$, $y = y^*$ then

$$f(x^*, y^*) = 0$$
$$g(x^*, y^*) = 0$$

$\cancel{x(t) = x^*}$

(x^*, y^*) - steady

$x(t=0) = x^*$, $x(t) = x^*$

$g(t=0) = y^*$ $y(t) = y^*$

$\forall t > 0$

(5) $\dot{\underline{x}} = \underline{f}(\underline{x}(t))$ $\quad \underline{x} = (x, y)$

$\underline{f} = \begin{pmatrix} f \\ g \end{pmatrix}$

$f(x^*, y^*) = g(x^*, y^*) = \emptyset$

$x(t) = x^* + u(t) \qquad |u(t)| << 1$

$y(t) = y^* + v(t) \qquad |v(t)| << 1$

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(x^* + u(t)\right) = \frac{d}{dt} u(t)$$

$$\frac{d}{dt} y(t) = \frac{d}{dt}\left(y^* + v(t)\right) = \frac{d}{dt} v(t)$$

$$\frac{d}{dt} u(t) = f(x^* + u(t), y^* + v(t))$$

$$= \underbrace{f(x^*, y^*)}_{=\emptyset} + \frac{\partial f(x^* y^*)}{\partial x} u(t) + \frac{\partial f(x^*, y^*)}{\partial y} v(t)$$

$$\frac{d}{dt} v(t) = g(x^* + u(t), y^* + v(t))$$

$$= \underbrace{g(x^*, y^*)}_{=\emptyset} + \frac{\partial g(x^* y^*)}{\partial x} u(t) + \frac{\partial g(x^* y^*)}{\partial y} v(t);$$

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \partial f(x^*, y^*)/\partial x & \partial f(x^* y^*)/\partial y \\ \partial g(x^* y^*)/\partial x & \partial g(x^* y^*)/\partial y \end{pmatrix}}_{\text{Jacobian}} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

$$f(x+\Delta) = f(x) + \Delta f'(x)$$

Examples

(1)

$$\dot{x}(t) = 3 - x - y - xy = f(x,y)$$
$$\dot{y}(t) = xy - 2y = g(x,y) = y(x-2)$$

$$f(x^*, y^*) = g(x^*, y^*) = 0$$

$$g(x,y) = y(x-2) = 0 \quad \begin{cases} y = 0 \Rightarrow 3 - x - 0 - 0 \cdot x = 0 \\ \qquad x = 3 \\ x = 2 \Rightarrow 3 - 2 - y - y2 = 1 - 3y = 0 \Rightarrow y = 1/3 \end{cases}$$

$x = 3,\ y = 0$ $\quad (x^*, y^*) = (3, 0)$

$x = 2 \quad y = 1/3$ $\quad (x^*, y^*) = (2, 1/3)$

EXAMPLE

$\boxed{\alpha \neq 0}$

$$\frac{d}{dt} u(t) = 1 - (1+\alpha) u + u^2 v = f(u, v)$$

$$\frac{d}{dt} v(t) = u - u^2 v = g(u, v) = u(1 - uv)$$

either $u = 0$ or $uv = 1$

Choice $u = 0$ first eq $f(u, v) = 1 \neq 0$

Second choice $uv = 1 \Rightarrow u = 1/v$

$$f(u, v) = 1 - (1+\alpha) \cdot \frac{1}{v} + \underbrace{(u \cdot v)}_{=1} \cdot u \qquad (u^*, v^*) = (1/\alpha, \alpha)$$

$$= 1 - \frac{(1+\alpha)}{v} + u = 0$$

$$= 1 - \frac{1+\alpha}{v} + \frac{1}{v} = 1 - \frac{1}{v} + \frac{1}{v} - \frac{\alpha}{v} = 0$$

$1 = \alpha/v$

$v = \alpha$

$u = 1/\alpha$

Example

$$\dot{x} = (x-1)(y-2)$$
$$\dot{y} = (x-3)(y-4)$$

First equation $x^* = 1$ or $y^* = 2$

$\Downarrow$ Second equation $\qquad \Downarrow$

$y^* = 4 \qquad x^* = 3$

$(x^*, y^*) = (1, 4) \qquad (x^*, y^*) = (3, 2)$

Example

$$\dot{x} = x - x^2 - xy = x(1-x-y) = f(x,y)$$

$$\dot{y} = 2y - y^2 - 3xy = y(2-y-3x) = g(x,y)$$

- Inspection: $x^* = y^* = \emptyset$
- Take $x^* = \emptyset$

$$f(x^* = \emptyset, y) = \emptyset$$

$$g(x^* = \emptyset, y) = y(2 - y - 3\cdot\emptyset)$$

$$y^* = 2$$

$(\emptyset,\emptyset)$
$(\emptyset,2)$
$(1,\emptyset)$
$(1/2,1/2)$
$= (x^* y^*)$

- Take $y^* = \emptyset \Rightarrow \dot{y} = \emptyset$ from second equation

$$\dot{x} = x(1-x) \Rightarrow x^* = 1$$

- $x + y = 1$

$3x + y = 2$

$2x = 1, \quad x = 1/2; \; y = 1/2$

$$\dot{R}(t) = R(t)\left(3 - R(t) - 2S(t)\right)$$
$$\dot{S}(t) = S(t)\left(2 - R(t) - S(t)\right)$$

Steady States:

- $R^* = S^* = \emptyset$

- $R^* = \emptyset$: $2 - R - S = \emptyset = 2 - S \Rightarrow (\emptyset, 2) = (R^*, S^*)$

- $S^* = \emptyset \Rightarrow 3 - R^* = \emptyset \Rightarrow R^* = 3$, $(R^*, S^*) = (3, \emptyset)$

- $$\begin{aligned} R + 2S &= 3 \\ R + S &= 2 \end{aligned} \Rightarrow S^* = 1 = R^*$$

$$(R^*, S^*) = (1, 1)$$

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$$f(R,S) = R(3-R-2S)$$

$$g(R,S) = S(2-R-S)$$

$\cancel{u = R^*}$

$$R(t) = R^* + u(t)$$

$$S(t) = S^* + v(t)$$

$\cancel{f(u,v)}$

$$J(R,S) = \begin{pmatrix} \partial f/\partial R & \partial f/\partial S \\ \partial g/\partial R & \partial g/\partial S \end{pmatrix} = \quad 3 - 2R - 2S$$

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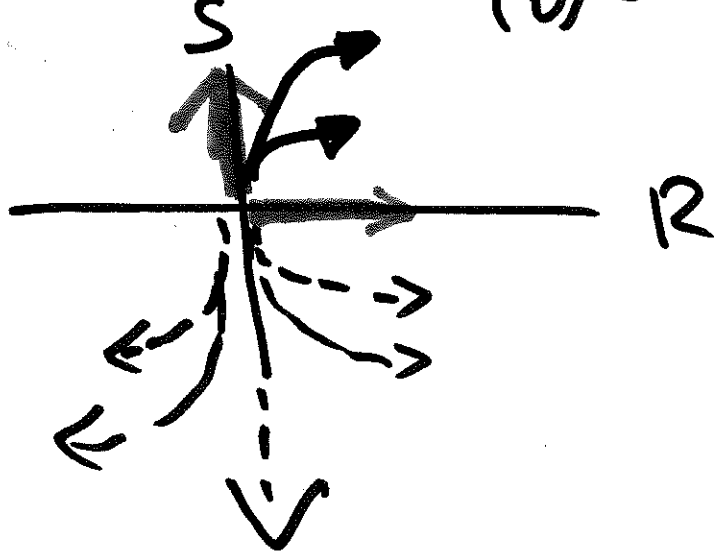
$$J(R,S) = \begin{pmatrix} 3-2R-2S & -2R \\ -S & 2-R-2S \end{pmatrix}$$

$R^* = S^* = \emptyset$: $J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \lambda_1 = 3;\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2;\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$(u,v) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$



- $R^* = S^* = 1$

$$J(R=1, S=1) = \begin{pmatrix} 3-2-2 & -2\cdot 1 \\ -1 & 2-1-2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \; ; \quad \begin{array}{l} Tr = -2 \\ \Delta = -1 \end{array}$$

$$\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4\Delta}}{2}$$

$$= \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$(-1-\lambda)(-1-\lambda) - 2 = 0$$

$$(\lambda+1)^2 = 2 \Rightarrow \cancel{\lambda}$$

$$\lambda + 1 = \pm\sqrt{2} \; ; \; \lambda = -1 \pm \sqrt{2}$$

$$-a - 2b = (-1-\sqrt{2})\,a$$

$$-2b = -\sqrt{2}\,a$$

$$\sqrt{2}\,b = a$$

$$\boxed{\begin{array}{c} v_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\ \lambda_1 = -1-\sqrt{2} \end{array}}$$

$$-c - 2d = (-1+\sqrt{2})\,c$$

$$-2d = \sqrt{2}\,c$$

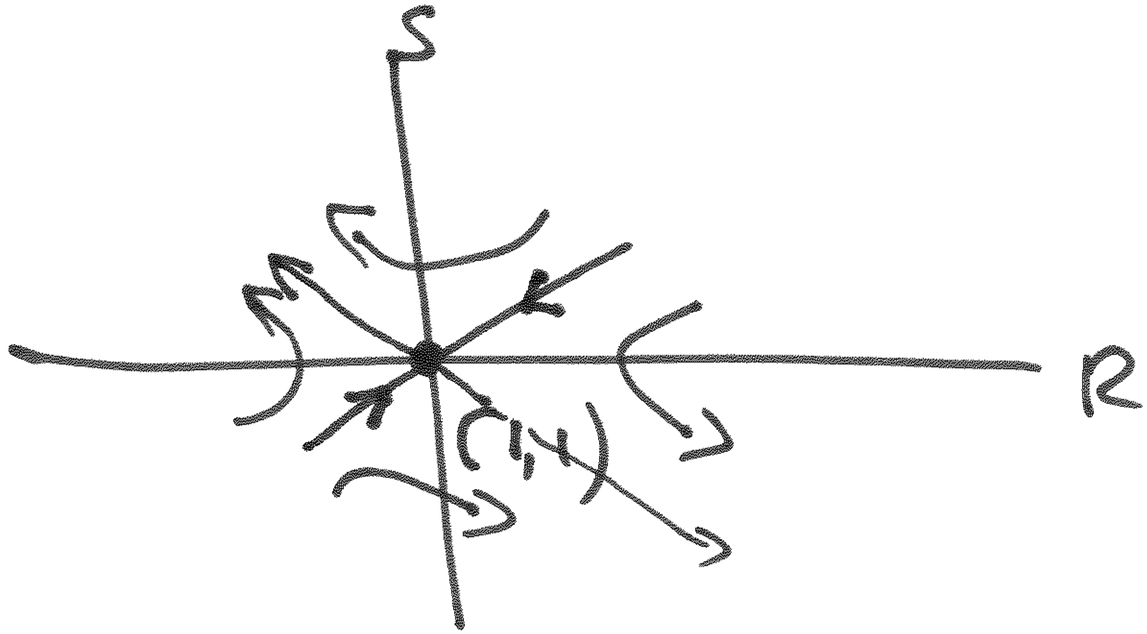
$$-\sqrt{2}\,d = c$$

$$v_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1+\sqrt{2}$$

$$(-1+\sqrt{2})t$$

$$\begin{pmatrix} R(t) \\ S(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_1 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-1-\sqrt{2})t} + C_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{\cancel{(-1+\sqrt{2})t}}$$


S
R
(1,4)

$$(R^*, S^*) = (3, 0)$$

$$J(3, 0) = \begin{pmatrix} 3-2\cdot 3 & -6 \\ 0 & 2-3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

$$(-3-\lambda)(-1-\lambda) = 0$$

$$(\lambda+3)(\lambda+1) = 0 \quad = \lambda_1 = -1$$

$$\lambda_2 = -3$$

$\boxed{\lambda_1 = -1}$

$$-3a - 6b = -a$$

$$2a + 6b = 0$$

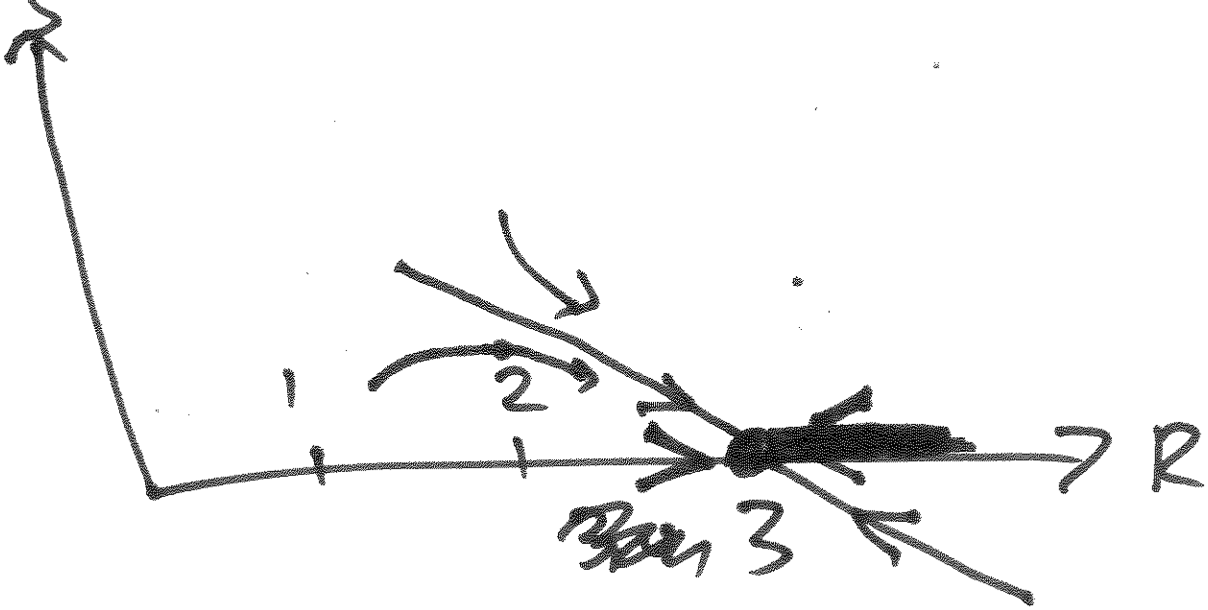
$$a + 3b = 0$$

$$v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$-3c - 6d = -3c$$

$$d = 0, \; c = 1;$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$


S
1
2
R
3

$$\bullet \; (R^*, S^*) = (\varnothing, 2)$$

$$J(\varnothing, 2) = \Big(\; R(3 - R - 2S)$$

$$J(\varnothing, 2) = \begin{pmatrix} 3 - 2R - 2S & -2R \\ -S & 2 - R - 2S \end{pmatrix}_{(\varnothing, 2)}$$

$$= \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \qquad \lambda_1 = -1, \quad \lambda_2 = -2$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; \lambda_1 = -1$$

$$\cancel{-a - \varnothing b = -a}$$

$$-2a - \cancel{2b} =$$

$$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$-a = -a, \quad a = 1$

$-2a - 2b = -b$

$-2a = b$

$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -2\begin{pmatrix} c \\ d \end{pmatrix}$$

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \lambda_2 = -2$

$-2c - 2d = -2d$

$c = \emptyset,$

$d = 1$