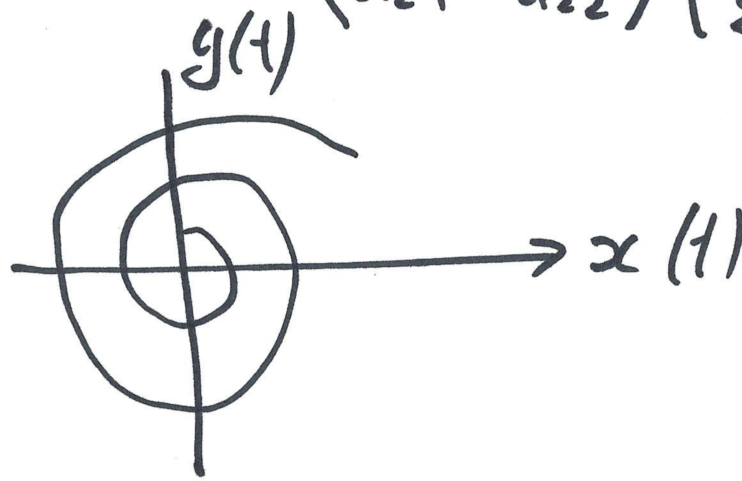
Nov 15 2024

Phase Plane Technique

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t))$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$



Finite # of possibilities.

Examples

$$\dot{x} = x \Rightarrow x(t) = x_0 e^{t}$$
$$\dot{y} = 2y \Rightarrow y(t) = y_0 e^{2t}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

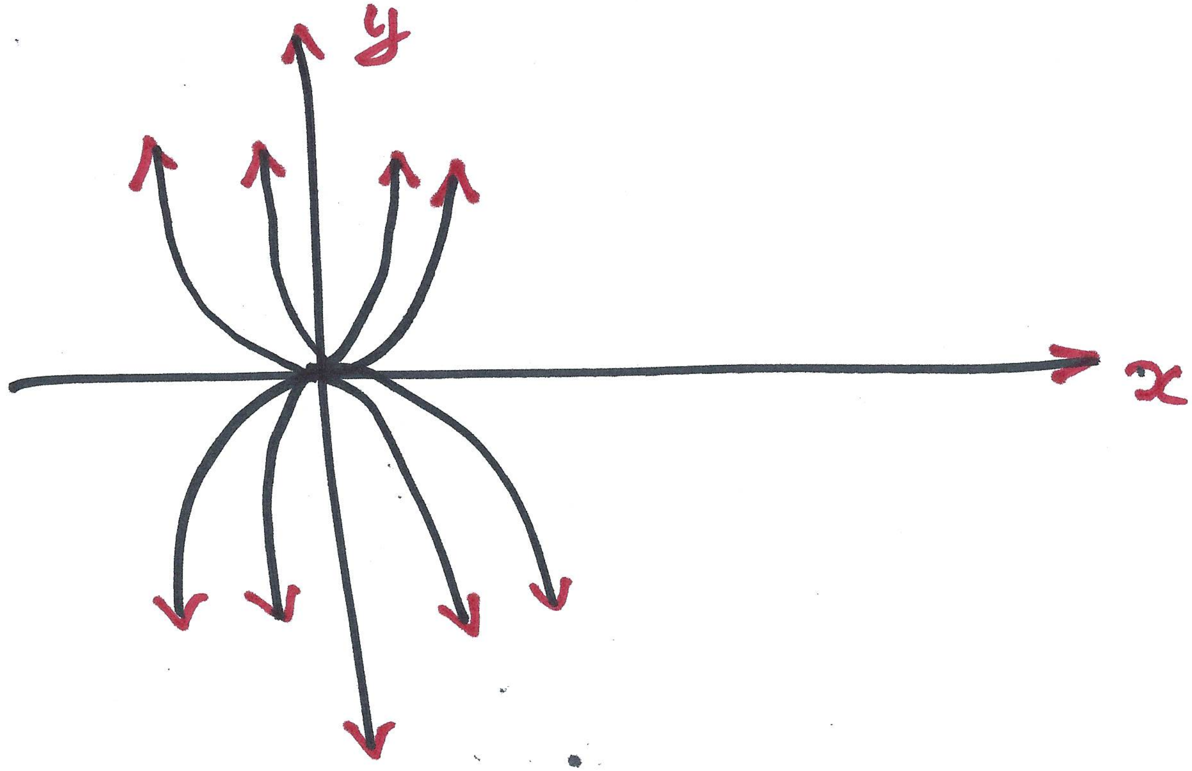
$$\underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix}; \quad \underline{\underline{A}}\,\underline{v}_1 = \lambda_1 \underline{v}_1$$

$$\underline{\underline{A}}\,\underline{v}_1 = 1 \cdot \underline{v}_1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left.\begin{array}{l} a = a \\ 2b = b \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a = 1 \\ b = 0 \end{array}\right.$$

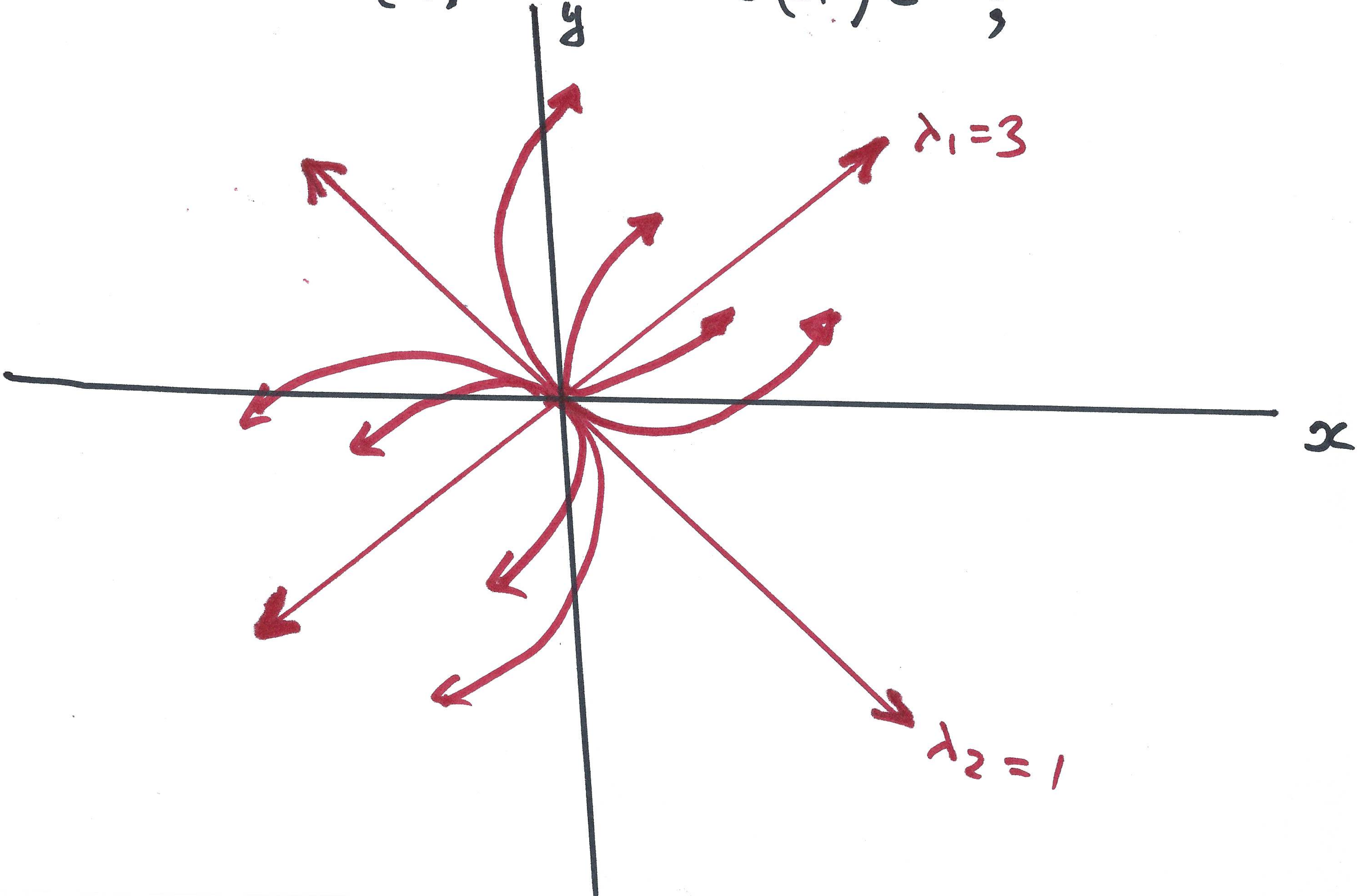
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

$$x(t) = c_1 e^{t} \Rightarrow e^{t} = x(t)/c_1$$
$$y(t) = c_2 e^{2t} \Rightarrow y(t) = \frac{c_2}{c_1^2} \cdot \left(x(t)\right)^2$$


y
x

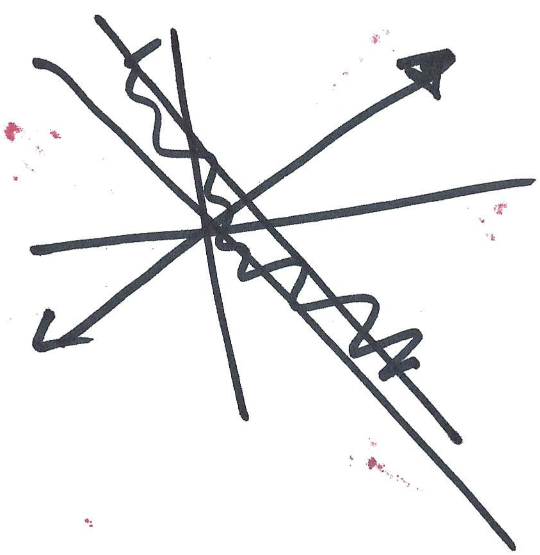
$$\underline{\dot{x}}(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \lambda_1 = 3;\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda_2 = 1;\ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t};$$



$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

A has two positive real roots



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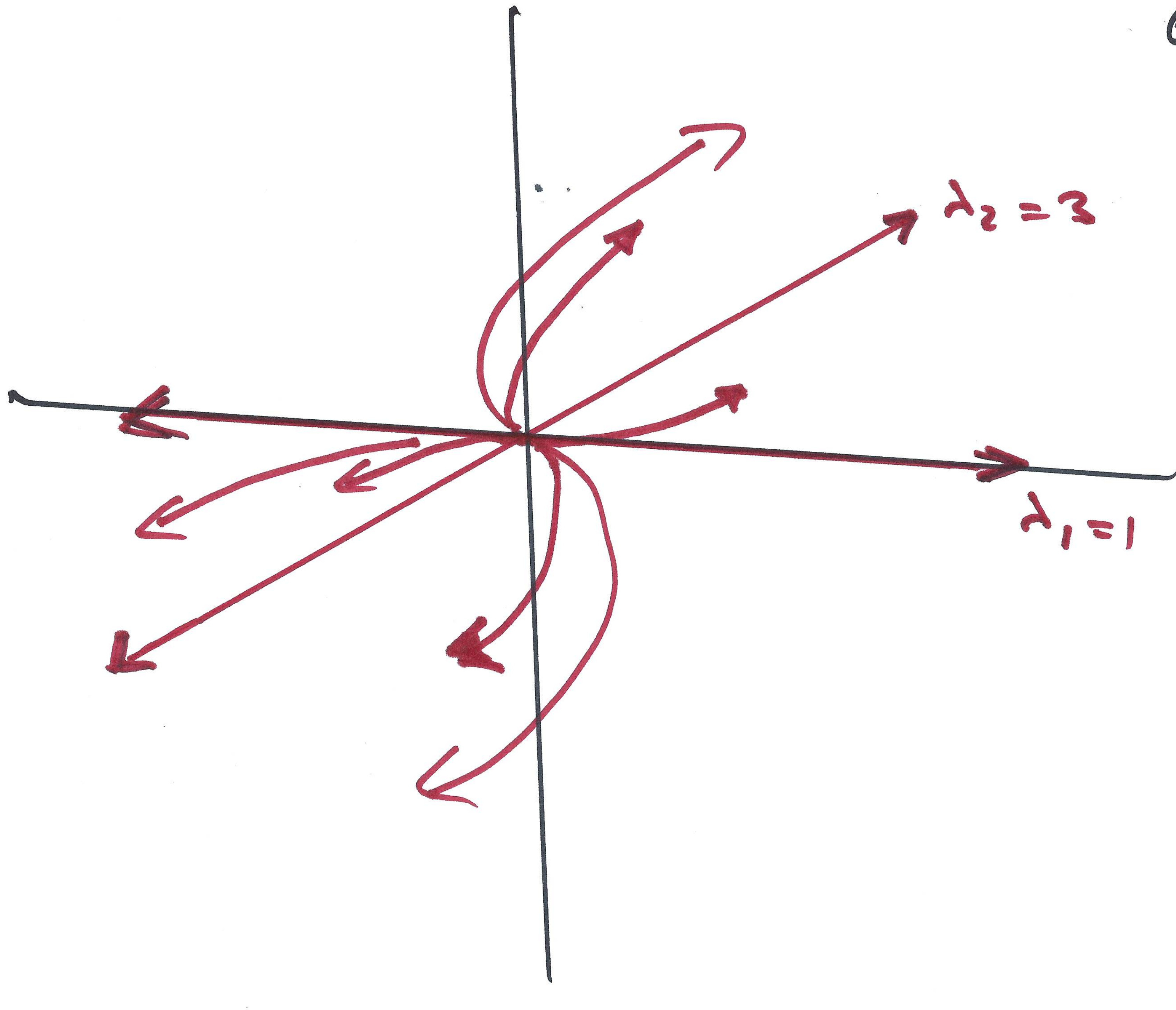
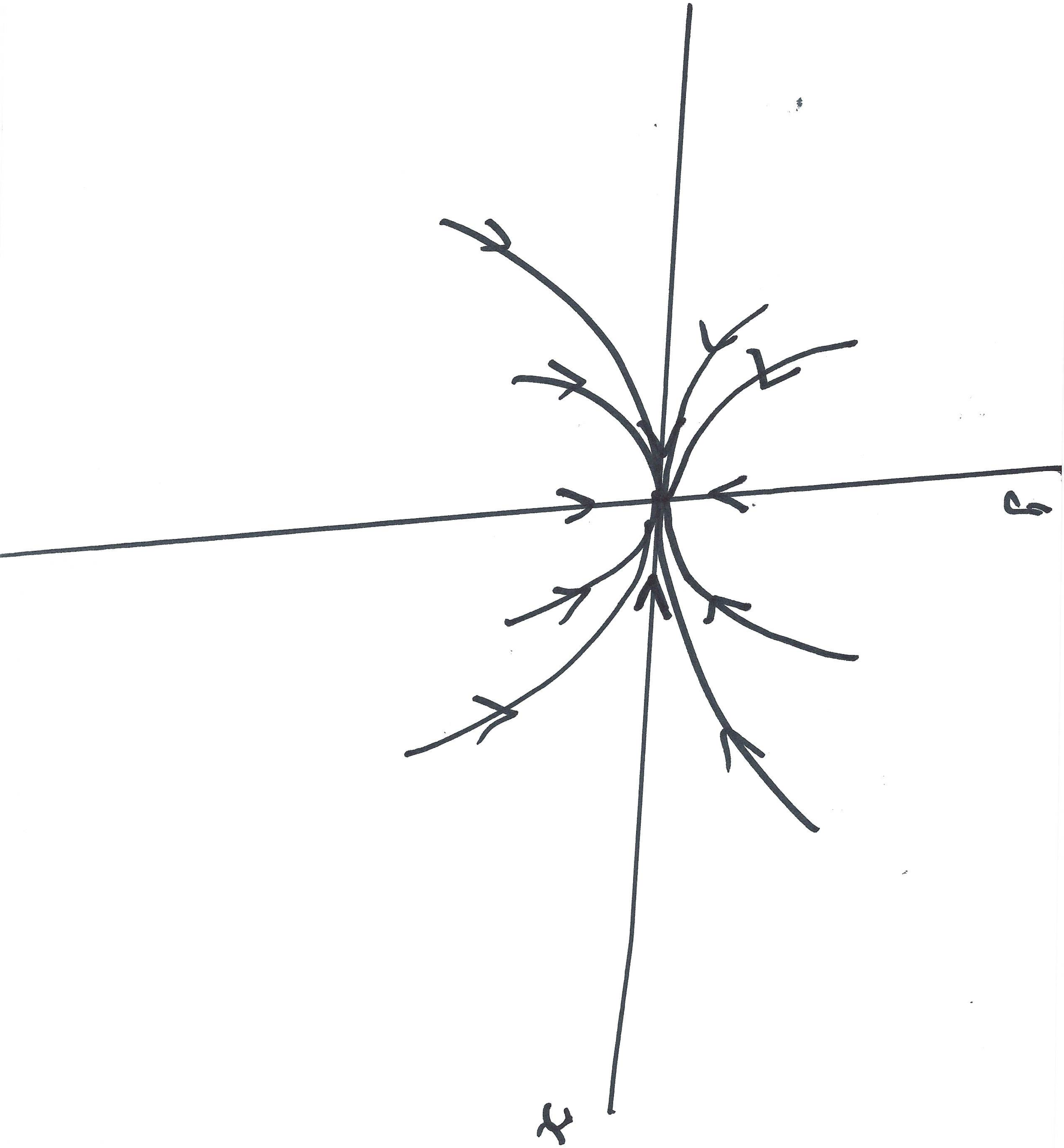


$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = -1; \ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = -2; \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

$$x(t) = c_1 e^{-t} \Rightarrow e^{-t} = x(t)/c_1$$

$$y(t) = c_2 e^{-2t} = \frac{c_2 \cdot (x(t))^2}{c_1^2}$$



$$\underline{\dot{x}} = \begin{pmatrix} -2 & 2 \\ 1/2 & -2 \end{pmatrix} \underline{x}$$

$$\emptyset = \mathrm{Det}(A - \lambda I) = (-2-\lambda)^2 - 2 \cdot 1/2 =$$

$$= (\lambda+2)^2 - 1 = \emptyset \Rightarrow \lambda + 2 = \pm 1$$

$$\lambda = \pm 1 - 2 = -3, -1$$

$\lambda_1 = -3$

$$\begin{pmatrix} -2 & 2 \\ 1/2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -3 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$-2a + 2b = -3a \Rightarrow -2b = +a \qquad v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$a/2 - 2b = -3b \Rightarrow a/2 = -b$$

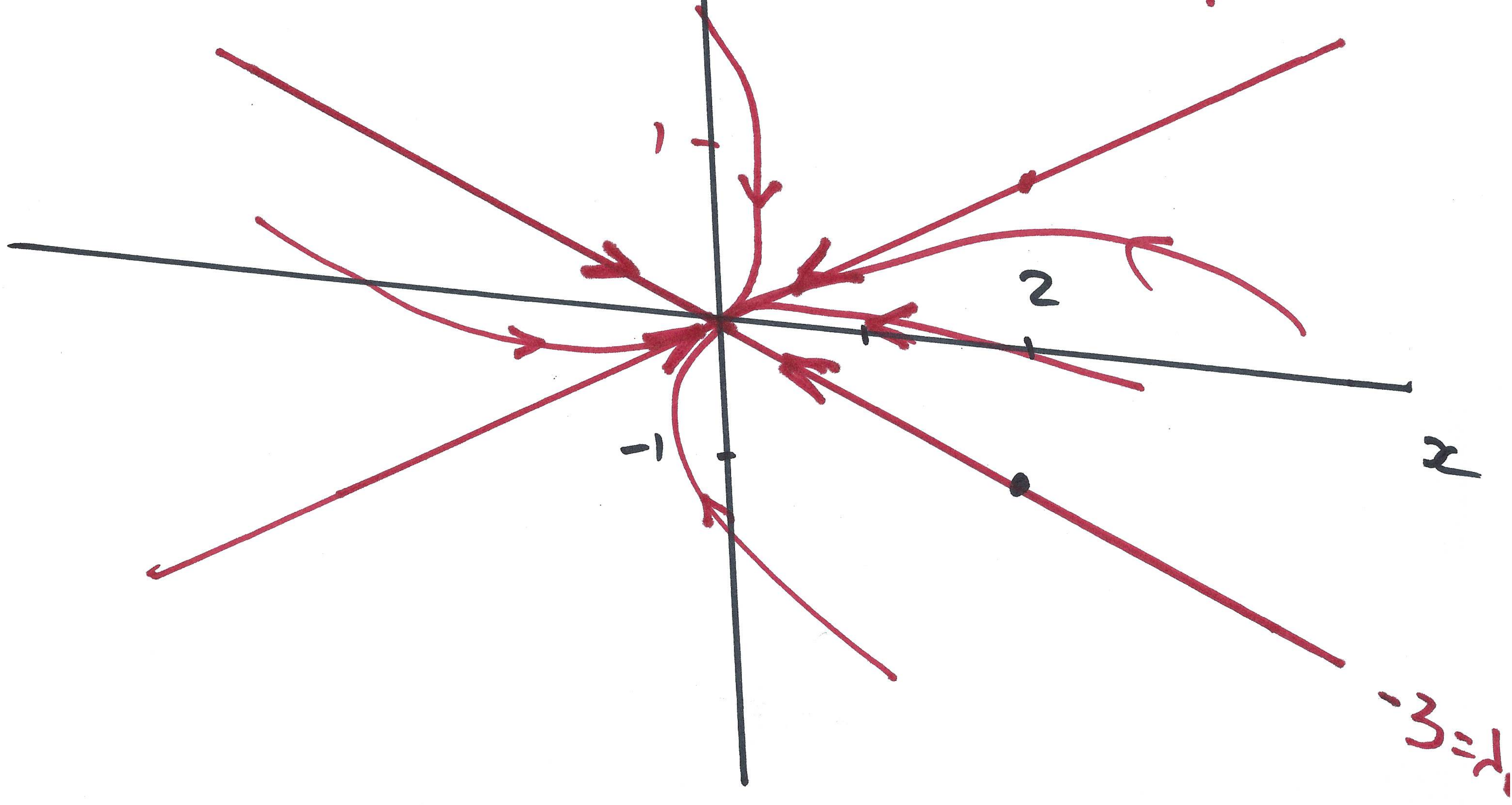
$$\begin{pmatrix} -2 & 2 \\ 1/2 & -2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -\begin{pmatrix} c \\ d \end{pmatrix}$$

$$-2c + 2d = -c \Rightarrow 2d = c$$

$$\frac{c}{2} - 2d = -d \Rightarrow \frac{c}{2} = d \Rightarrow c = 2d$$

$$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$(\emptyset, \emptyset)$ is stable fixed point



Example

$$\underline{\dot{x}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underline{x}(t)$$

$\lambda_1 = 1$ $\quad$ $\lambda_2 = -1$

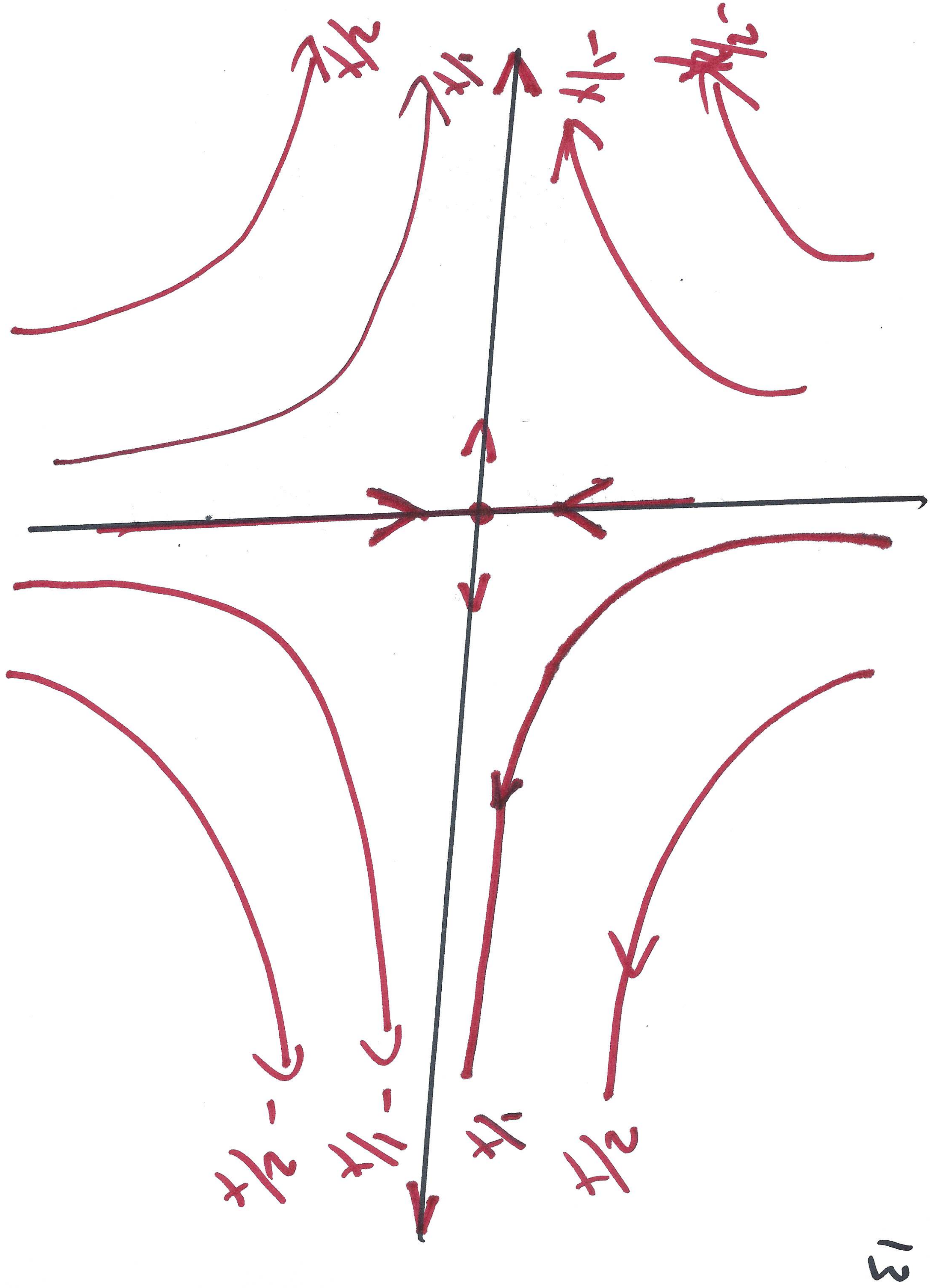
$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$x(t) = c_1 e^{t} \Rightarrow e^{t} = x/c_1$$

$$y(t) = c_2 e^{-t} = \frac{c_2}{x/c_1} = \frac{c_1 c_2}{x}$$

$x \equiv x(t)$



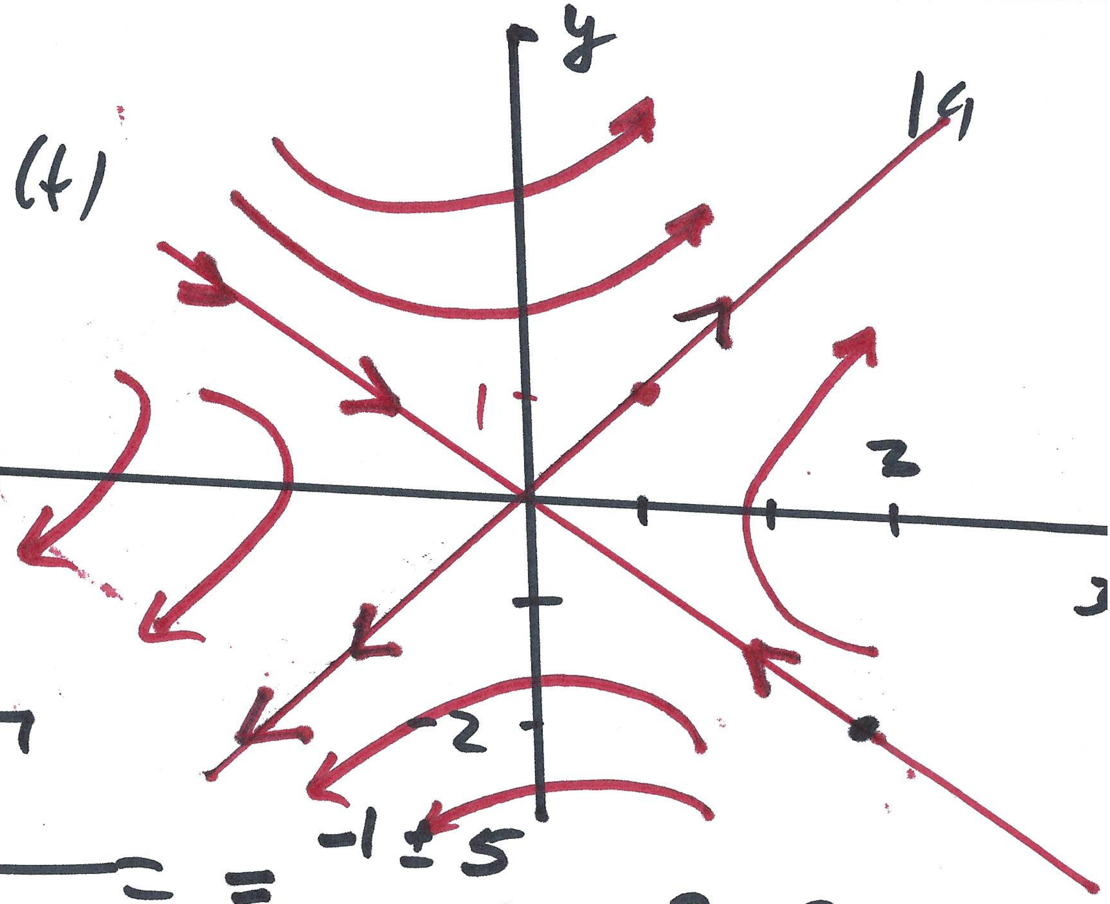
$$\dot{\underline{x}}(t) = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \underline{x}(t)$$

$$(-1-\lambda)(-\lambda) - 6 = \emptyset$$

$$(\lambda+1\cdot)\lambda - 6 = \emptyset$$

$$\lambda^2 + \lambda - 6 = \emptyset$$

$$\lambda_{12} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = -3;\ 2;$$



(-3) $\quad -a + 3b = -3a \Rightarrow 3b = -2a$

$\quad 2a = -3b \qquad v_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(2) $\quad -c + 3d = 2c \Rightarrow d = c$

$\quad 2c = 2d \Rightarrow c = d \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = 2$$

$$\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$2a = 2a \qquad a = a$$

$$2b = 2b \qquad b = b$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 25 \\ 23 \end{pmatrix}$$

$$\underline{x}(t) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \qquad v_2 = \begin{pmatrix} 1 \\ 147 \end{pmatrix}$$

$$x(t) = c_1 e^{2t}$$

$$y(t) = c_2 e^{2t} = \frac{c_2}{c_1} x(t)$$

Two vectors are linearly independent

if

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 = \emptyset$$

$$\Updownarrow \quad \text{iff "if and only if"}$$

$$c_1 = c_2 = \emptyset$$

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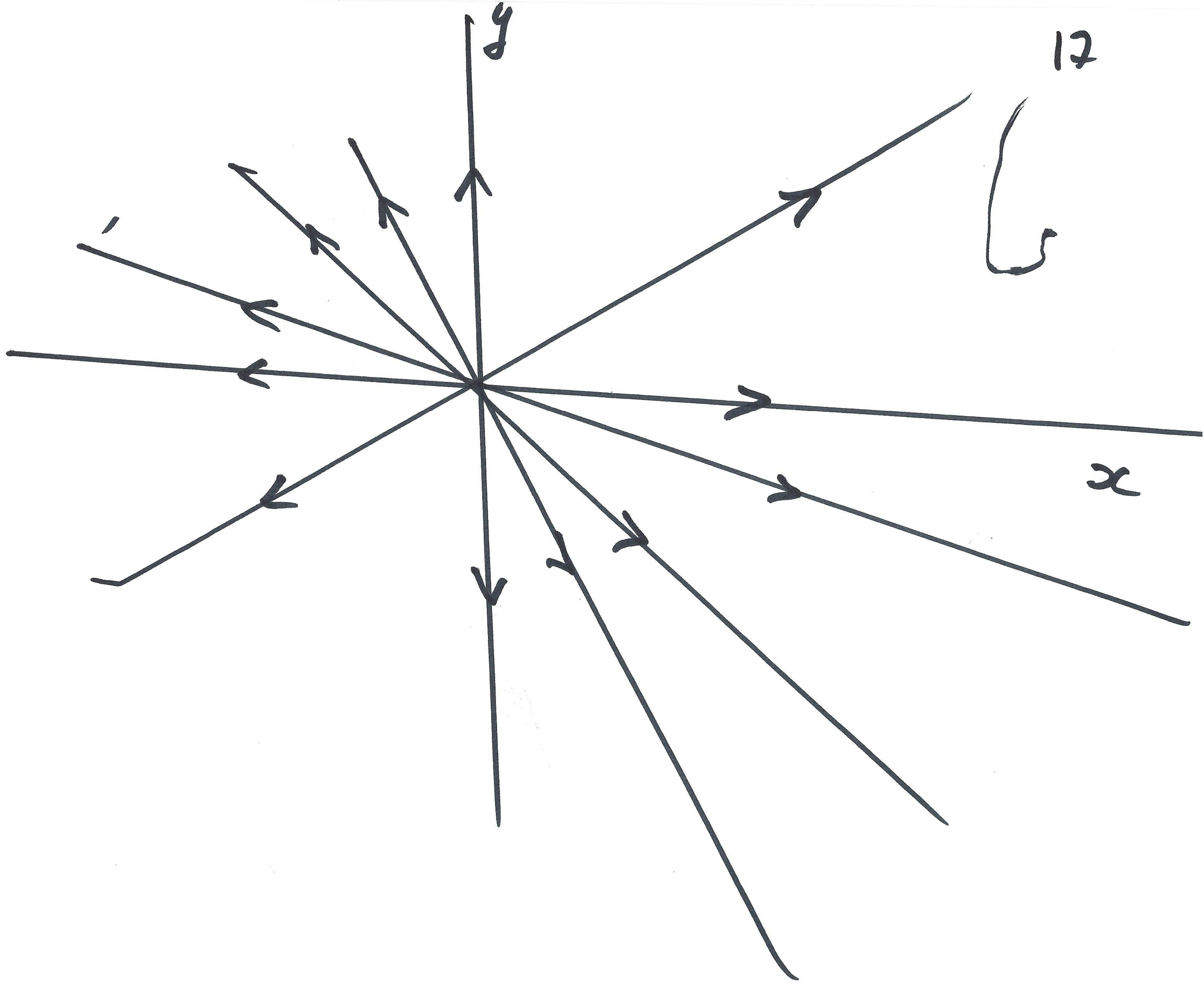

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$$\underline{\dot{x}}(t) = \begin{pmatrix} -25 & 0 \\ 0 & -25 \end{pmatrix} \underline{x}(t)$$