$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}; \quad \left.\begin{aligned} \frac{d}{dt}\underline{x}(t) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underline{x}(t) \\ \underline{x}(t=0) &= \underline{x}_0 \end{aligned}\right\} \text{IVP}$$

$\underline{x}(t)$ is a vector function

$x(t)$, $y(t)$ are scalar function

$$\begin{cases} \dfrac{dx(t)}{dt} = a_{11}\, x(t) + a_{12}\, y(t) \\[2ex] \dfrac{dy(t)}{dt} = a_{21}\, x(t) + a_{22}\, y(t) \end{cases}$$

$$x(t=0) = x_0$$

$$y(t=0) = y_0$$

$R(t) \quad J(t)$

$R(t) > 0$ R loves J

$R(t) < 0$ R hates J

$J(t) > 0$ love

$J(t) < 0$ hate

$$\begin{cases} \frac{d}{dt} R(t) = a_{11} R(t) + a_{12} J(t) \\ \frac{d}{dt} J(t) = a_{21} R(t) + a_{22} J(t) \end{cases}$$

are

J, R is unaware of his feeling

their

$$\frac{d}{dt} R(t) = J(t)$$

$$\frac{d}{dt} J(t) = -R(t)$$

$$\frac{d}{dt}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix}$$

$$\frac{d}{dt}\,\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

Find eigenpair (eigenvector and eigenvalue) of A

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda I \underline{x} = \varnothing$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \varnothing \Rightarrow \mathrm{Det}(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \varnothing$$

quadratic equation

$\text{Det}\,(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

has 2 solutions

→ Two solutions, both are real

Find two eigenvectors $\underline{v}_1, \underline{v}_2$

$$\underline{x}(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} \quad (*)$$

→ one double solution,

It will be real,

(a) There are two distinct eigenvectors

Then the solution is $(*)$

Two real eigenvalues that coincide
and only ONE eigenvector

A is "deffective"

Look for solution as

$$\underline{x}(t) = (\underline{v}\, t + \underline{a})\, e^{\lambda t} \qquad (**)$$

Find $\underline{a}$ such that is a solution

$$\emptyset = \dot{\underline{x}}(t) - \underline{\underline{A}}\,\underline{x} =$$

$$= \left(\underline{v} + \lambda(\underline{v}t + a) - \underbrace{\underline{\underline{A}}\,\underline{v}}_{\lambda \underline{v}}\, t - \underline{\underline{A}}\,\underline{a}\right) e^{\lambda t}$$

cancel

$$\underline{v} + \lambda \underline{a} - \underline{\underline{A}}\,\underline{a} = \varnothing$$

$$-\underline{v} - \underline{\lambda}\underline{a} + \underline{\underline{A}}\,\underline{a} = \varnothing$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{a} = \underline{v}$$

$$\underline{x}(t) = c_1 \underline{v} e^{\lambda t} + c_2 (\underline{v} t + \underline{a}) e^{\lambda t}$$

$$\frac{d}{dt}\underset{\sim}{x}(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \underset{\sim}{x}(t);$$

Find eigenvalues

$$\text{Det}(A - \lambda I) = \emptyset$$

$$\text{Det}\left[\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] =$$

$$= (2-\lambda)^2 = \emptyset \Rightarrow \lambda = 2$$

$$\underline{v} = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = 2 \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

$$\left.\begin{array}{r} 2\mu + \nu = 2\mu \\ 2\nu = 2\nu \end{array}\right] \Rightarrow \nu = \emptyset \quad \mu = \text{any}$$

$\downarrow$ ν is anything

$$\underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{a} = \underline{v}$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda = 2$$

$$\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \underline{a} = \begin{pmatrix} u \\ w \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} u \cdot 0 + w = 1 \\ 0 \cdot u + 0 w = 0 \end{array}\right]$$

$$w = 1, \; u = 0.$$

$$a = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\underline{x}(t) = c_1 \underline{v} e^{\lambda t} + c_2 (\underline{v} t + \underline{a}) e^{\lambda t} \quad \boxed{\lambda = 2}$$

$$= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{2t}$$

$$= \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{2t}$$

$W = 1$, $\cancel{u = \alpha}$; $u = 23$

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 23 \\ 1 \end{pmatrix} \right] e^{2t}$$

$$= \begin{pmatrix} c_1 + c_2 t + 23 c_2 \\ c_2 \end{pmatrix} e^{2t}$$

Solve $\det(A - \lambda I) = \emptyset$ and you get 2 complex conjugate eigenvalues.

Find complex eigenvector

Eigenvalues are complex conjugate of each other

Eigenvectors — " — " — " —

$\lambda = \gamma + i\omega$ $\quad \gamma = \operatorname{Re}\lambda$ $\quad \underline{a} = \operatorname{Re}\underline{v}$

$\underline{v} = \underline{a} + i\underline{b}$ $\quad \omega = \operatorname{Im}\lambda$ $\quad \underline{b} = \operatorname{Im}\underline{v}$

$$\underline{x}(t) = D_1(\underline{a}\cos\omega t - \underline{b}\sin\omega t)e^{\gamma t}$$
$$+ D_2(\underline{a}\sin\omega t + \underline{b}\cos\omega t)e^{\gamma t}$$

$$\frac{d}{dt}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

$$\mathrm{Det}\,(A - \lambda I) = \mathrm{Det}\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} =$$

$$= \lambda^2 + 1 = \emptyset$$

$$\lambda^2 = -1 \Rightarrow \lambda = \pm i$$

$$\underline{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$v = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} b = ia \\ -a = ib \end{matrix} \Rightarrow b = ia$$

$\underline{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$; $\underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\lambda = \pm i$

$\gamma = \emptyset, \omega = 1$

$$\underline{x}(t) = D_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right)$$

$$+ D_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right)$$

$$= \begin{pmatrix} D_1 \cos t + D_2 \sin t \\ -D_1 \sin t + D_2 \cos t \end{pmatrix} = \begin{pmatrix} R(t) \\ J(t) \end{pmatrix}$$

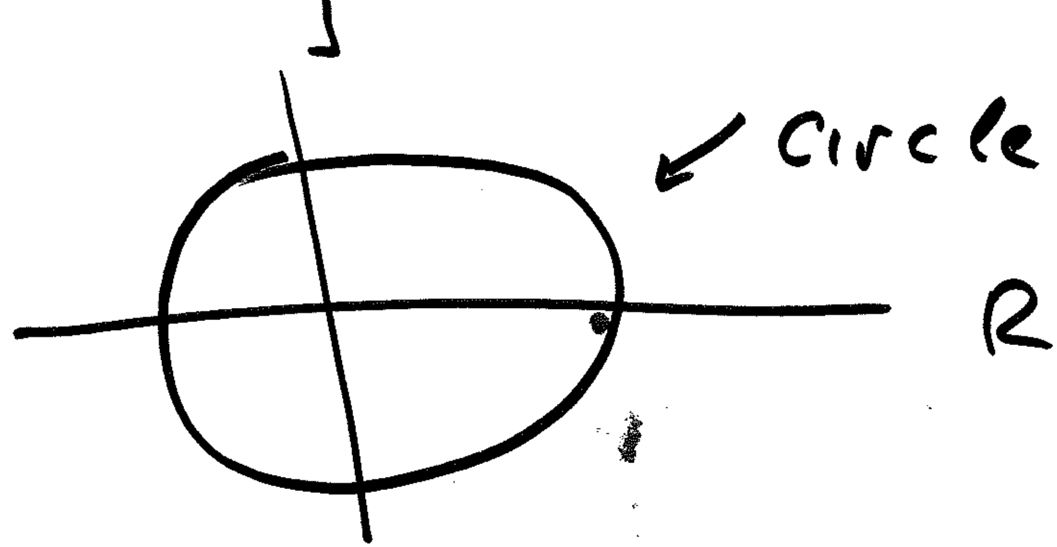
$$R^2(t) + J^2(t) = (D_1 \cos t + D_2 \sin t)^2$$

$$+ (-D_1 \sin t + D_2 \cos t)^2 =$$

$$=$$

$$R^2 + J^2 = \underbrace{D_1^2 \cos^2 t + D_1^2 \sin^2 t}_{= D_1^2} + \underbrace{D_2^2 \sin^2 t + D_2^2 \cos^2 t}_{D_2^2} + \underbrace{2 D_1 D_2 \cos t \sin t - 2 D_1 D_2 \cos t \sin t}_{0}$$

$$= D_1^2 + D_2^2$$



$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}\,, \quad \underline{x}(t=0) = \underline{X}_0$$

① Find eigenvalues $\operatorname{Det}(A - \lambda I) = 0$

② Find eigenvectors solve $\underline{\underline{A}}\,\underline{v}_1 = \lambda \underline{v}_1$

and $\underline{\underline{A}}\,\underline{v}_2 = \lambda v_2$

③ write the general solution

→ Real $\lambda_1 \neq$ Real λ_2

$$\underline{X}(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} \quad (*)$$

→ Real $\lambda_1 =$ Real λ_2

2 eigenvectors (λ)

1 eigenvector (deffective)

$(A - \lambda I)\underline{a} = \underline{v}$

$$\underline{x}(t) = C_1 v e^{\lambda t} + C_2 (v t + a) e^{\lambda t}$$

complex λ_1.

$$\lambda_1 = \lambda_2^*$$

$\lambda = \gamma + i\omega$

$v = a + ib$

$$x(t) = \left[D_1(a\cos\omega t - b\sin\omega t) + D_2(a\sin\omega t + b\cos\omega t)\right] e^{\gamma t}$$

4 Satisfy Initial condition

find C_1, C_2

$$\frac{d}{dt}\underline{x}(t) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \underline{x}(t)$$

$$x(\emptyset) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

① Eigenvalues

$$\text{Det}\begin{pmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix} = \emptyset \Rightarrow \lambda(\lambda-1) - 2 = \emptyset$$

$$\lambda^2 - \lambda - 2 = \emptyset$$

$$\lambda_{12} = \frac{1 \pm 3}{2} = -1, 2$$

$\cancel{v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad v_1}$

(-1) $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} b = -a \\ 2a + b = -b \end{matrix}; \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\underline{x}(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

$$x(t=\emptyset) =$$

$$= \begin{pmatrix} C_1 + C_2 \\ -C_1 + 2C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C_1 = -C_2$$

$$3C_2 = 1,$$

$$x(t) = -\frac{1}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} \quad C_2 = 1/3$$

- $R = 1;\ I = \emptyset;$

$$R = D_1 \cos t + D_2 \sin t$$

$$I = -D_1 \sin t + D_2 \cos t$$

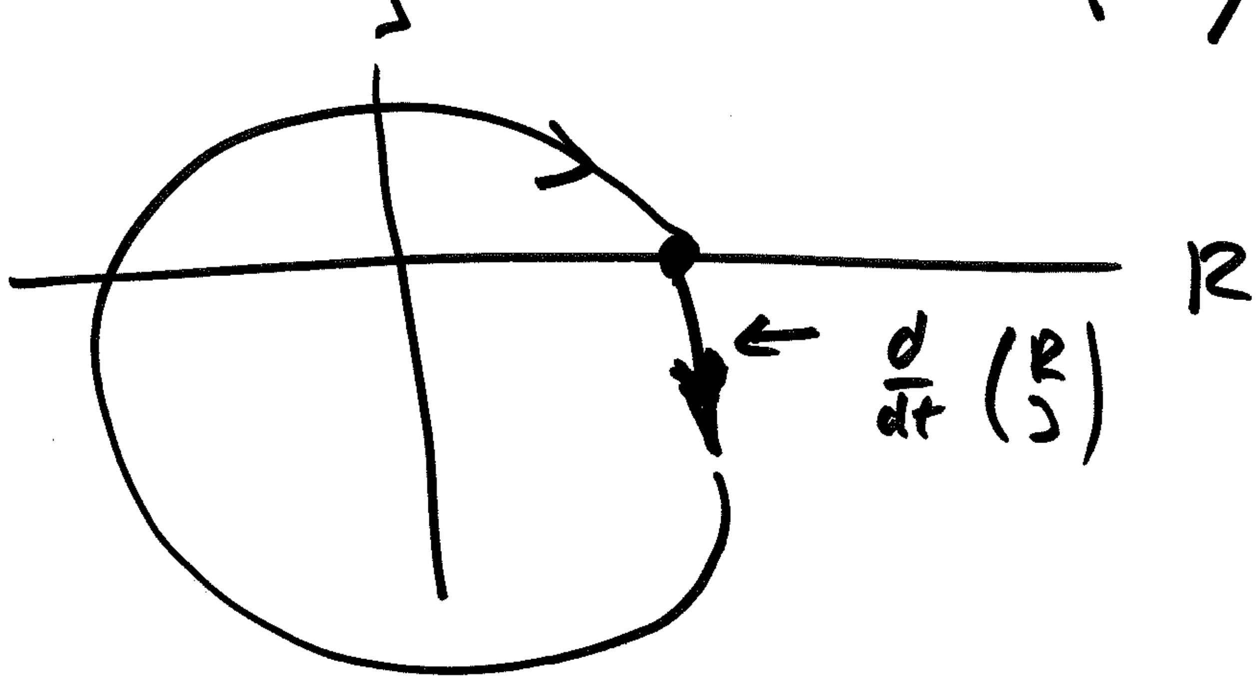
$\underline{t = \emptyset}$, $R = D_1 = 1$ $\quad R(t) = \cos t$

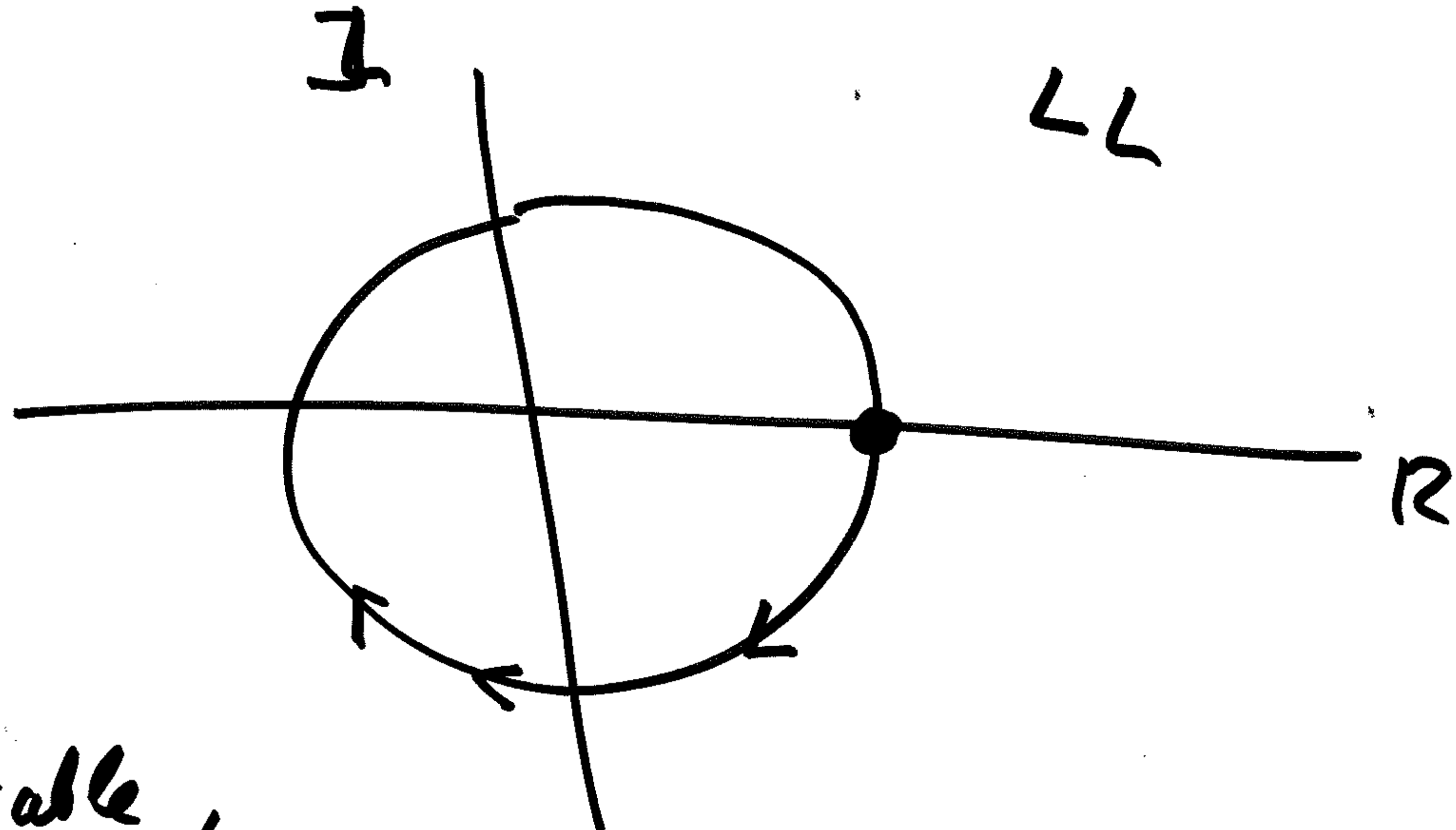
$I = D_2 = \emptyset$ $\quad I = -\sin t$

$R = 1, \; J = \emptyset$

$$\frac{d}{dt}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} R(t) \\ J(t) \end{pmatrix}$$

$$t = \emptyset, \quad \frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$





Stable,

mutual half the time

love-love quarter of the time