Integrating factor Method

Linear Equation 1st order

$$y'(x) + p(x)y(x) = g(x)$$

* $\mu(x)$ - integrating factor

$$\mu(x)y'(x) + p(x)\mu(x)y(x) = \mu(x)g(x)$$

LHS: $\frac{d}{dx}\left(\mu(x)y(x)\right) = \mu(x)y'(x) + p(x)\mu(x)y(x)$

$$\mu'(x)y(x) + \underbrace{\mu(x)y'(x)} = \underbrace{\mu(x)y'(x)} + p(x)\mu(x)y(x);$$

$$\mu'(x)y(x) = p(x)\mu(x)y(x); \quad y(x) \neq 0$$

$$\mu'(x) = p(x)\mu(x);$$

$$\mu'(x) = \mu(x)\, p(x);$$

$$\int \frac{d\mu(x)}{\mu(x)} = \int p(x)\, dx$$

$$\ln \mu(x) = \cancel{e^{\int p(x)}} \int p(x)\, dx$$

$$\mu(x) = e^{\int p(x)\, dx};$$

$$y'(x) + p(x)\,y(x) = g(x) \quad \Big] \quad \times\, e^{\int p(x)\,dx}$$

$$\frac{d}{dx}\, y(x)\, e^{\int p(x)\,dx} + p(x)\,y(x)\,e^{\int p(x)\,dx}$$

$$= g(x)\, e^{\int p(x)\,dx}$$

$$\frac{d}{dx}\left(y(x)\, e^{\int p(x)\,dx} \right) = g(x)\, e^{\int p(x)\,dx}$$

$$\frac{d}{dx}\left(y(x)\,\mu(x)\right) = g(x)\,\mu(x)$$

$$y(x)\,\mu(x) = \int g(x)\,dx \cdot \mu(x);$$

D

$$\boxed{y(x) = \frac{1}{\mu(x)} \int g(x)\, \mu(x)\, dx}$$

$$\boxed{\mu(x) = e^{\int p(x)\, dx}}$$

$y'(x) + p(x)\, y(x) = g(x)$;

$p(x)$, $g(x)$ are given

$$\mu(x) = \int dx \; p(x)$$

Examples:

$$y'(x) + y(x) = 1; \qquad p(x) = 1$$
$$g(x) = 1$$

$$y'(x) + p(x)\,y(x) = g(x)$$

$$\cancel{\mu(x) = e}$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int dx} = e^{x}$$

$$\left(y'(x) + y(x) = 1\right)e^{x}$$

$$y'(x)e^{x} + y(x)e^{x} = e^{x}$$

$$\int\left(\frac{d}{dx}\left(y(x)e^{x}\right) = e^{x}\right)dx$$

$$y(x)e^{x} = \int e^{x}\,dx = e^{x} + C$$

$$y(x) = 1 + Ce^{-x}$$

<u>Check:</u> $\frac{d}{dx}\left(y(x)\right) = -Ce^{-x}$

$$y' + y - 1 = -Ce^{-x} + 1 + Ce^{-x} - 1 = \varnothing$$

$y'(x) + y(x) = 1$ – non homogeneous

$y(x) = 1 + Ce^{-x}$

corresponding homogeneous equation

$y'(x) + y(x) = \emptyset$

$y(x) = Ce^{-x}$ – general solution to homogeneous equation

$y(x) = 1$

General solution of a linear nonhomogeneous equation is a sum of

general solution of corresponding homogeneous equation

AND

a particular solution of nonhomogeneous equation.

I

$$y'(x) + 2y(x) = e^{-x}$$

$$p(x) = 2; \quad g(x) = e^{-x};$$

$$\mu(x) = e^{\int 2dx} = e^{2x}$$

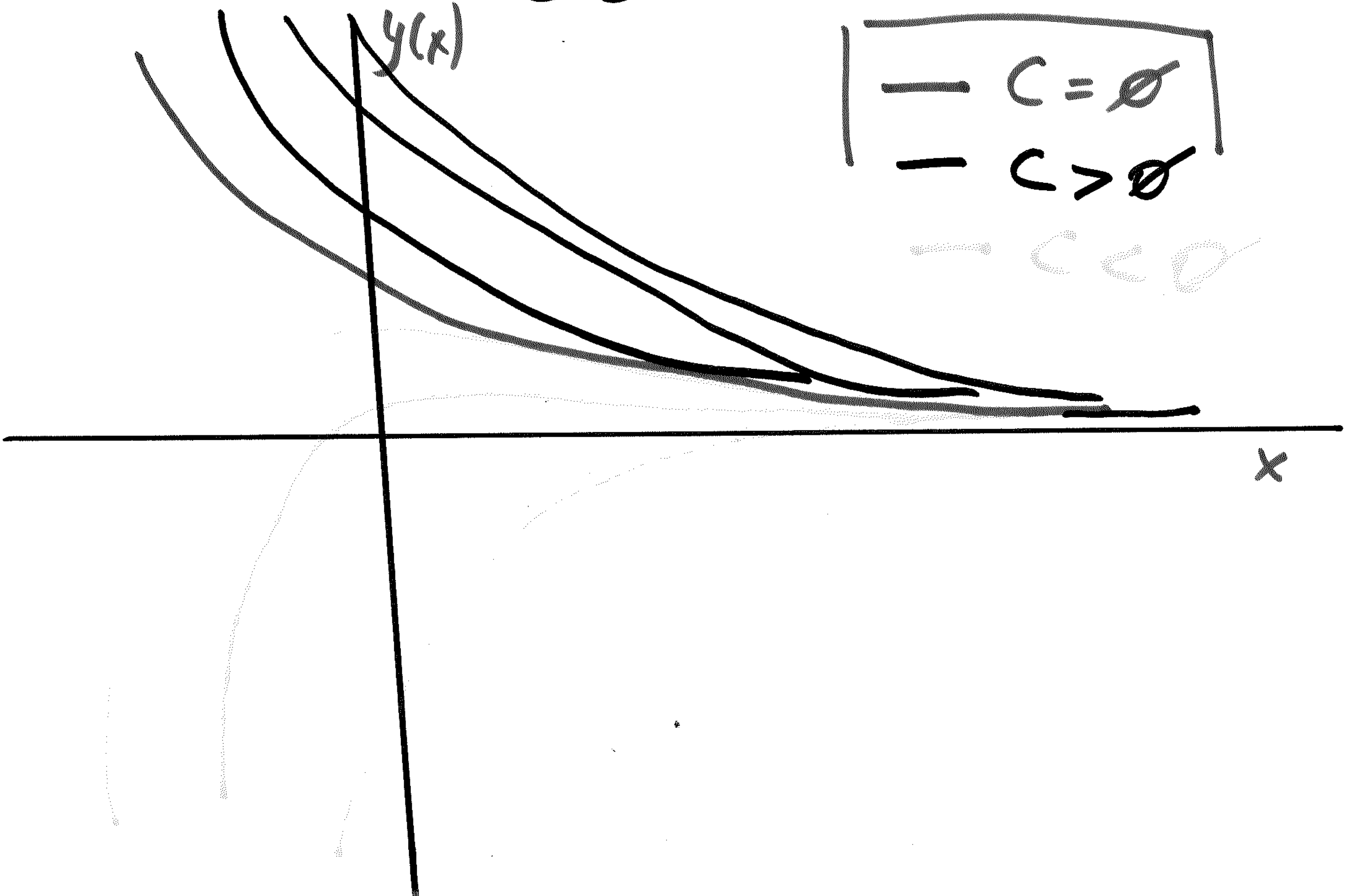
$$y'(x)\, e^{2x} + 2y(x)e^{2x} = e^{x}$$

$$\int \left(\frac{d}{dx}\left(y(x)e^{2x}\right) = e^{x} \right) dx$$

$$y(x)\, e^{2x} = \int e^{x}\, dx = e^{x} + C$$

$$y(x) = e^{-x} + Ce^{-2x}$$

$$y(x) = e^{-x} + Ce^{-2x}$$



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F

$$\left(y'(x) + 2y(x) = e^{-x}\right)$$

$$\mu(x) = e^{\int 2\,dx} = e^{2x+C}$$

$$e^{2x+C}\, y'(x) + 2e^{2x+C}\, y(x) = e^{-x} e^{2x+C}$$

$$\frac{d}{dx}\left(y(x)\, e^{2x+C}\right) = e^{+x+C}$$

$$y(x)\, e^{2x+C} = e^{x+C} + C_2$$

$$y(x) = e^{-x} + C_2 e^{-2x-C}$$

$$= e^{-x} + \boxed{C_2 e^{-C}} \cdot e^{-2x}$$

$$y'(x) - 2x\,y = x;$$

$$y(0) = 0$$

$$p(x) = -2x; \quad g(x) = x;$$

$$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$$

$$y'(x)\,e^{-x^2} - 2x\,e^{-x^2} = x\,e^{-x^2}$$

$$\frac{d}{dx}\left(y(x)\,e^{-x^2}\right) = x\,e^{-x^2}$$

$$u = x^2$$

$$y(x)\,e^{-x^2} = \int x\,e^{-x^2}\,dx =$$

$$= \int \frac{du}{2} e^{-u} = -\frac{1}{2} e^{-u} + C$$

I

$$y(x)\,e^{-x^2} = -\frac{1}{2}e^{-x^2} + C$$

$$y(x) = -\frac{1}{2} + C\,e^{x^2}$$

$$y(\emptyset) = -\frac{1}{2} + C = \emptyset \implies C = 1/2$$

$$y(x) = -\frac{1}{2} + \frac{1}{2}e^{x^2} = \frac{1}{2}\left(e^{x^2} - 1\right)$$

$$\emptyset \overset{?}{=} y'(x) - 2xy - x$$

$$= x\,e^{x^2} - 2x\left(-\frac{1}{2} + \frac{1}{2}e^{x^2}\right) - x$$

$$= xe^{x^2} + x - xe^{x^2} - x = \emptyset$$

J

$$t\,y'(t) + 2y(t) = 9t^2$$

K

$$t\,y'(t) + 2\,y(t) = 4t^2$$ Not a standart form

$$y(1) = 2;$$

X $$\mu(t) = e^{\int p(x)\,dx} = e^{\int 2\,dt} = e^{2t}$$

$$y'(t) + \frac{2}{t}\,y(t) = 4t;$$

$$\cancel{\mu(t)} = e$$

$$\mu(t) = e^{\int \frac{2\,dt}{t}} = e^{2\int \frac{dt}{t}} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2\,y'(t) + 2t\,y(t) = 4t^3$$

$$\frac{d}{dt}\left(t^2\,y(t)\right) = t^2 y'(t) + 2t\,y(t) = 4t^3$$

L

$$t^2 y(t) = \int 4t^3\,dt = t^4 + C$$

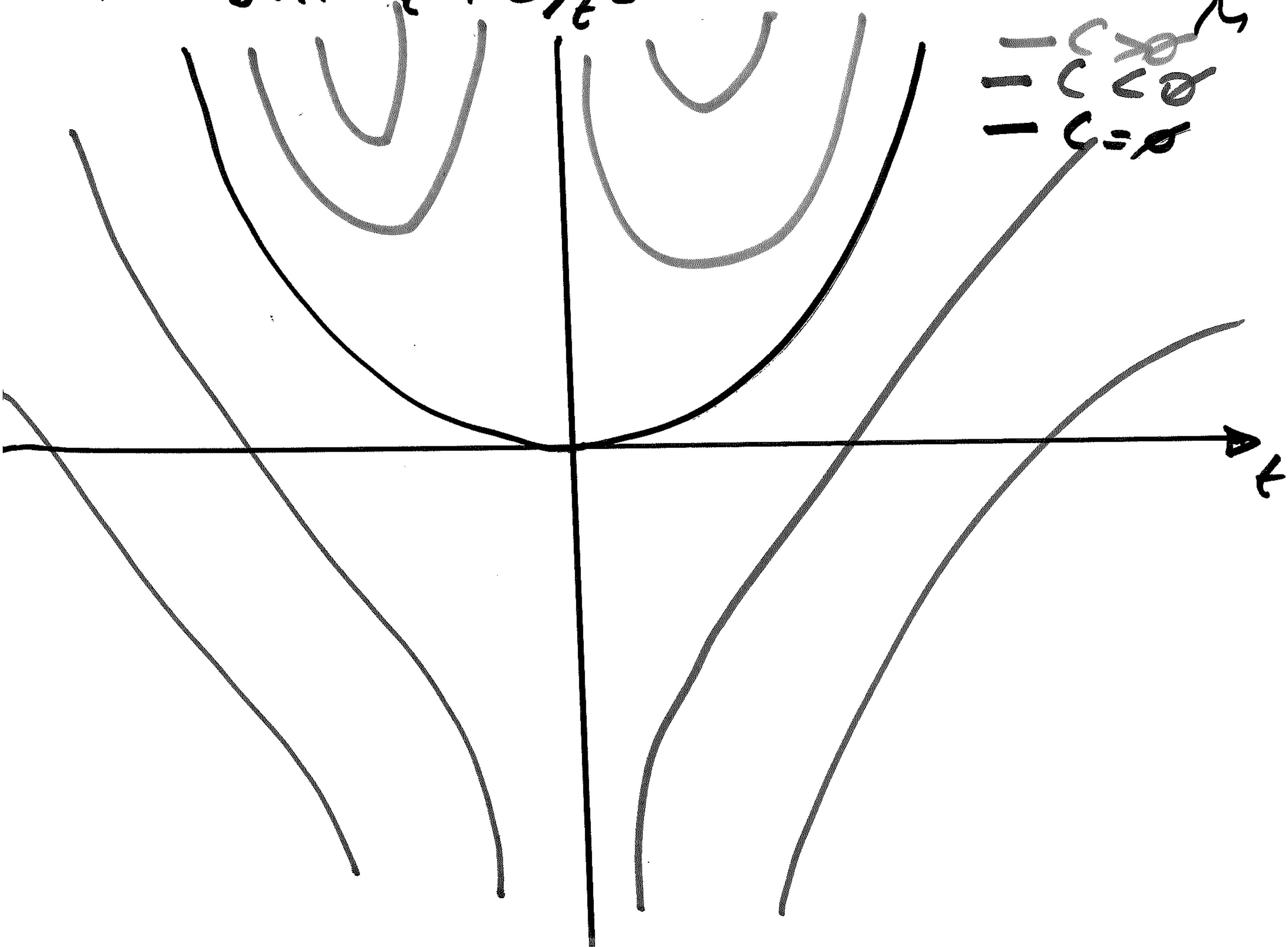
$$y(t) = t^2 + C/t^2$$ - general sltn

~~$y(1) = t^2 +$~~

$$y(1) = 1 + C/1 = 2 \Rightarrow C = 1$$

$$y(t) = t^2 + 1/t^2$$

$y(t) = t^2 + c/t^2$



N

$$y'(x) - 2xy(x) = 1$$

$p(x) = -2x;$

$g(x) = 1;$

$$y(x) = e^{x^2}\left(C + \int_0^x e^{-x^2}\,dx\right)$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2}$$

$$y'(x)e^{-x^2} - 2xe^{-x^2}y(x) = e^{-x^2}$$

$$\frac{d}{dx}\left(y(x)e^{-x^2}\right) = e^{-x^2}$$

$$y(x)e^{-x^2} = \int e^{-x^2}\,dx; \quad y(x) = e^{+x^2}\int e^{-x^2}\,dx$$

0

$$y'(x) + p(x)y(x) = g(x)$$

$$\mu(x) = e^{\int p(x)\,dx}$$

$$\frac{d}{dx}\left(y(x)\mu(x)\right) = g(x)\mu(x)$$

$$y(x) = \frac{1}{\mu(x)} \int g(x)\,dx$$

P

Population growth:

$$\frac{d}{dt} \text{Quantity}(t) = \text{Stuff In} - \text{Stuff Out}$$

$$P(t)$$

$$\frac{d\,P(t)}{dt} = k\,P(t), \quad k > 0$$

$$P(t=0) = P_0$$

$$\int \frac{d\,P(t)}{P(t)} = \int k\,dt = kt + C = \ln P$$

$$P(t) = e^{kt+C} = e^{kt} e^{C} = P_0 e^{kt}$$

$$P(t) = P_0 e^{kt}$$

Time to double?

$$P(t=0) = P_0$$

$$P(t=T) = P_0 e^{kT} = 2P_0$$

$$e^{kT} = 2; \quad kT = \ln 2$$

$$\boxed{T = \frac{\ln 2}{k}}$$