Eigenvector: $\underline{\underline{A}} \cdot \underline{x} = \lambda \underline{x};$

eigenvalue

eigenvector

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow$ $\lambda_1 = 1$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \lambda_2 = 2$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\lambda_1 = 1 \quad \lambda_2 = 1$

2d
Any vector is an eigenvector

$v_1 =$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$v_1' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\mathrm{Det}(A - \lambda I) = (2-\lambda)^2 - 1 = 0$$

$$\cancel{\lambda_{1,2}} = 1$$

$$(\lambda - 2)^2 = 1$$

$$\lambda - 2 = \pm 1$$

$$\lambda = 2 \pm 1 = 3, 1$$

$\lambda_1 = 1$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 3$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{v_2} = \begin{pmatrix} a \\ b \end{pmatrix}$: $A v_2 = 3 v_2$

$2a + b = 3a, \quad a = b$

$a + 2b = 3b \Rightarrow a = b$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$; $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$\lambda = 1$ Solve $\underline{\underline{A}}\,\underline{v_1} = 1 \cdot \underline{v_1}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{array}{l} 2a + b = a \\ a + 2b = b \end{array}\right\}$$

$a = -b$

$-a = 1 = b$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$v_2 = \begin{pmatrix} -25 \\ 25 \end{pmatrix}$

Example

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) =$$

$$= \det\left[\begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} 2-\lambda & 4 \\ -1 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 + 4 = 0$$

$$(\lambda - 2)^2 = -4$$

$$\lambda - 2 = \pm 2i \qquad \lambda = 2 \pm 2i$$

$$\left.\begin{array}{l}\lambda_1 = 2+2i \\ \underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix};\end{array}\right\}$$

$$\boxed{\begin{array}{l}\lambda_2 = 2-2i \\ v_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}\end{array}}$$

$$\underline{\underline{A}} \cdot \underline{v}_1 = \lambda_1 \underline{v}_1$$

$$\begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (2+2i)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$2a + 4b = 2a + 2i\,a \Rightarrow 4b = 2i\,a \Rightarrow 2b = ia$$

$$-a + 2b = (2+2i)b \Rightarrow -a = 2i\,b \Rightarrow 2b = ia$$

$$2b = ia$$

$$a = -2i\,b$$

$$b = 1$$

$$\boxed{\begin{array}{l}v_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix} \\ \lambda_1 = 2+2i\end{array}}$$

IF $A = A^* \Rightarrow A \in \mathbb{R}^{n \times n}$

$$Av = \lambda v$$

$$(Av)^* = (\lambda v)^*$$

$$Av^* = \lambda^* v^*$$

Example

$$A = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$$ deffective

$\lambda_1 = 5 = \lambda_2$

$$\det(A - \lambda \underline{I}) =$$

$$= \det\begin{pmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 = \emptyset$$

$$\underline{v}_1 = \begin{pmatrix} c \\ d \end{pmatrix} : \quad A v_1 = \lambda_1 v_1$$

$$\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 5\begin{pmatrix} c \\ d \end{pmatrix}$$

$5c + d = 5c \Rightarrow d = \emptyset$

$5d = 5d \Rightarrow d = d \Rightarrow d$ is any number

$v_1 = \begin{pmatrix} c \\ \emptyset \end{pmatrix} \Rightarrow$

$c = 1$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\frac{d}{dt} x_1 = a_{11} x_1 + a_{12} x_2$$

$$\frac{d}{dt} x_2 = a_{21} x_1 + a_{22} x_2 \qquad \Rightarrow x_1 \equiv x_1(t), \quad x_2 \equiv x_2(t)$$

$$\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$$

where $(\lambda_1, \underline{v}_1)$ and (λ_2, v_2) are eigenpairs of $\underline{\underline{A}}$.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$\lambda_1 = 1$ $\qquad\qquad$ $\lambda_2 = 3$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\qquad\qquad$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t};$$

$$x(t) = c_1 e^{t} + c_2 e^{3t}$$
$$y(t) = -c_1 e^{t} + c_2 e^{3t}$$

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$$\dot{x} = 2x + y$$

$$\alpha = -\dot{x} + 2x + y$$

$$= \underline{-c_1 e^t} - \underline{\underline{3c_2 e^{3t}}} + \underline{2c_1 e^t} + 2\underline{\underline{c_2}} e^{t\cdot 3}$$

$$- \underline{c_1 e^t} + \underline{\underline{c_2 e}}^{t\cdot 3} = \emptyset \checkmark$$

$$\beta = -\dot{y} + x + 2y = c_1 e^t - c_2 e^{3t} \cdot 3$$

$$+ c_1 e^t + c_2 e^{3t}$$

$$-2c_1 e^t + 2c_2 e^{3t} = \emptyset$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\underline{x}(t=\emptyset) = \begin{pmatrix} x(t=\emptyset) \\ y(t=\emptyset) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\left.\begin{aligned} x(t=\emptyset) &= c_1 + c_2 = \emptyset \\ y(t=\emptyset) &= -c_1 + c_2 = 2 \end{aligned}\right\} \quad \begin{aligned} c_1 &= -c_2 \\ c_2 &= 1 \\ c_1 &= -1 \end{aligned}$$

$$x(t) = -e^{t} + e^{3t}$$

$$y(t) = e^{t} + e^{3t}$$

$$\dot{\underline{x}}(t) = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix} \underline{x}(t)$$

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\frac{d}{dt} x(t) = 2x(t) + 4y(t)$$

$$\frac{d}{dt} y(t) = -x(t) + 2y(t)$$

$$\lambda_1 = 2 + 2i$$

$$v_1 = \begin{pmatrix} -2i \\ +1 \end{pmatrix}$$

$$\lambda_2 = 2 - 2i$$

$$v_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\underline{x}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \qquad e^{ix} = \cos x + i \sin x$$

$$= c_1 \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(2+2i)t} + c_2 \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(2-2i)t}$$

$$= c_1 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{2t} \left(\cos(2t) + i \sin(2t) \right)$$

$$+ c_2 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{2t} \left(\cos(2t) - i \sin(2t) \right)$$

$$= e^{2t} \Big[c_1 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right)$$

$$+ c_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right)$$

$$+\, i\left[\left(-\begin{pmatrix}2\\0\end{pmatrix}\cos 2t + \begin{pmatrix}0\\1\end{pmatrix}\sin 2t\right)C_1\right.$$

$$\left. + \left(-\begin{pmatrix}2\\0\end{pmatrix}\cos 2t + \begin{pmatrix}0\\1\end{pmatrix}\sin 2t\right)C_2\right]$$

$$= \overbrace{\boxed{(C_1+C_2)}}^{D_1} e^{2t}\left(\begin{pmatrix}0\\1\end{pmatrix}\cos 2t + \begin{pmatrix}2\\0\end{pmatrix}\sin 2t\right)$$

$$+ \underbrace{\boxed{i(C_1-C_2)}}_{D_2} e^{2t}\left(\begin{pmatrix}2\\0\end{pmatrix}\cos 2t + \begin{pmatrix}0\\1\end{pmatrix}\sin 2t\right)$$

$$\underline{\dot{x}} = \underline{\underline{A}} \cdot \underline{x} \, ;$$

$$\underline{\underline{A}}\,\underline{v} = \underline{\lambda}\,\underline{v}$$

$$\lambda = \gamma + i\omega \qquad \gamma = \operatorname{Re}\lambda$$

$$\underline{v} = \underline{a} + i\underline{b} \qquad \omega = \operatorname{Im}\lambda$$

$$\underline{a} = \operatorname{Re}\underline{v}$$

$$\underline{b} = \operatorname{Im}\underline{v}$$

$\underline{x}($

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$$\underline{x}(t) = c_1 \underline{v} e^{\lambda t} + c_2 \underline{v}^* e^{\lambda^* t}$$

$$\cancel{= c_1 (\underline{a} + i \underline{b}) e^{\gamma t}}$$

$$= c_1 (\underline{a} + i \underline{b}) e^{(\gamma + i\omega)t} + c_2 (\underline{a} - i\underline{b}) e^{(\gamma - i\omega)t}$$

$$= c_1 (\underline{a} + i\underline{b}) e^{\gamma t} \left(\cos(\omega t) + i \sin(\omega t)\right)$$
$$+ c_2 (\underline{a} - i\underline{b}) e^{\gamma t} \left(\cos(\omega t) - i \sin(\omega t)\right)$$

$$= c_1 (\underline{a} \cos \omega t - \underline{b} \sin \omega t) e^{\gamma t}$$
$$D_1 \quad + c_2 (\underline{a} \cos \omega t - \underline{b} \sin \omega t) e^{\gamma t}$$
$$+ i c_1 (\underline{a} \sin \omega t + \underline{b} \cos(\omega t)) e^{\gamma t}$$
$$D_2 \quad - i c_2 (\underline{a} \sin \omega t + \underline{b} \cos(\omega t)) e^{\gamma t}$$

$$\underline{x}(t) = \Big[D_1\big(\underline{a}\cos(\omega t) - \underline{b}\sin(\omega t)\big) + D_2\big(\underline{a}\sin(\omega t) - \underline{b}\cos(\omega t)\big)\Big]e^{\gamma t}$$

$$\underline{a} = \mathrm{Re}\,\underline{v}$$

$$\underline{b} = \mathrm{Im}\,\underline{v}$$

$$\gamma = \mathrm{Re}\,\lambda$$

$$\omega = \mathrm{Im}\,\lambda$$

Deffective matrices

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$$

A is "deffective"

$$(A - \lambda I)\underline{b} = \underline{v}$$

$$\underline{x}(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + \cancel{???} \; (\underline{v} t + b) e^{\lambda_1 t}$$

$$\underline{x}(t) = e^{\lambda t} (\underline{v} \cdot t + \underline{b})$$

$$\underline{\dot{x}}(t) = e^{\lambda t} (\lambda \underline{v} t + \lambda \underline{b} + \underline{v})$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}} e^{\lambda t} (\underline{v} t + \underline{b})$$

$$= e^{\lambda t} (\lambda \underline{v} t + \underline{\underline{A}}\,\underline{b})$$