$\lambda_1 = 3, \lambda_2 = -2$

$(\lambda - 3)(\lambda + 2) = \varnothing;$

$\lambda^2 - 3\lambda + 2\lambda - 6 = \varnothing$

$\lambda^2 - \lambda - 6 = \varnothing$

$y''(x) - y'(x) - 6y(x) = \varnothing;$

$y(x) = C_1 e^{3x} + C_2 e^{-2x};$

$y'(x) = 3C_1 e^{3x} - 2C_2 e^{-2x};$

Single ODE 2nd order linear => system of 2 1st order linear ode

$$\left.\begin{aligned} z(x) &= y'(x) \\ z'(x) &= z(x) + 6y(x) \end{aligned}\right\}$$

$$\begin{cases} y'(x) = z(x) \\ z'(x) = 6y(x) + z(x) \end{cases}$$

$$\frac{d}{dx}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} \emptyset & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix}$$

$$\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3x} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2x};$$

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_1 = 3 \qquad \lambda_2 = -2$$

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}; \quad \underline{\underline{A}} \cdot \underline{v_1} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\underline{\underline{A}} \cdot \underline{v_2} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Let $\underline{\underline{A}} \in R^{n \times n}$, then

$\underline{v}$ is called an eigenvector

λ is called an eigenvalue

IF

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

$$\underline{\underline{A}}\,\underline{v} - \lambda \underline{v} = \emptyset$$

$$\underline{\underline{A}}\,\underline{v} - I \lambda \underline{v} = \emptyset$$

I is a unity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

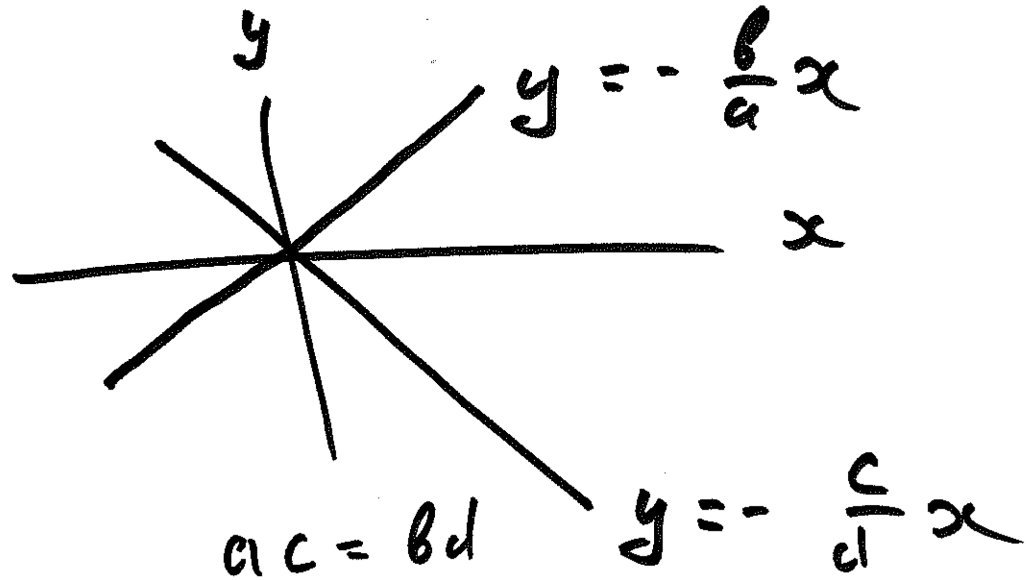
$$(\underline{\underline{A}} - \underline{\underline{I}}\lambda)\underline{v} = \emptyset$$

. Detour $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\left[\begin{array}{l} ax + by = \emptyset \\ cx + dy = \emptyset \end{array}\right.$$

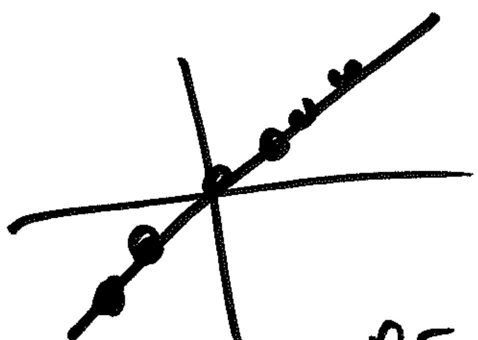
when $(x, y) \neq \emptyset$

$$\left.\begin{array}{l} y = -\frac{b}{a}x \\ y = -\frac{c}{d}x \end{array}\right\}$$



$\Downarrow$ If $-\frac{b}{a} = -\frac{c}{d}$ $\quad ac = bd$

then



two lines coincide

or Determinant $= ac - bd = \emptyset$

Want $(\underline{\underline{A}} - \lambda \underline{\underline{I}})v = \emptyset$

possible IFF

$$\text{Det}(A - \lambda I) =$$

$$= \text{Det}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \text{Det}\left(\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} \right) =$$

$$= (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = \emptyset$$

Ex

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$$

Eigenvalues: $\mathrm{Det}(A - \lambda I) = \emptyset$

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{pmatrix}$$

$$\mathrm{Det}(A - \lambda I) = -\lambda(1-\lambda) - 6 =$$

$$= \lambda^2 - \lambda - 6 = \emptyset$$

$$\lambda_1 = -2; \quad \lambda_2 = +3$$

If $A \in R^{n \times n}$

then $\text{Det}(A - \lambda I)$ is
a polynomial of degree n

Main theorem of Algebra
Polynomial of degree n
has n roots

$A \in R^{n \times n}$ it has n eigenvalues

$A \in R^{n \times n}$

it has n eigenvalues:

To find eigenvectors:

For each eigen value

Solve $\underline{\underline{A}}\,\underline{v}_i = \lambda_i \underline{v}_i$

eigenvector

eigenvalue

$(\lambda_i, \underline{v}_i)$ - eigenpair

Example

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$$

$$\lambda_1 = 3; \quad \lambda_2 = -2$$

Eigenvectors:

$\lambda_1 = 3$; solve $\underline{\underline{A}}\,\underline{v}_1 = \lambda_1 \underline{v}_1$

$$\begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \underline{v}_1 = 3 \cdot v_1$$

$$\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 3\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$v_{12} = 3 v_{11}$ ⟵ same ⟶

$$6 v_{11} + v_{12} = 3 v_{12} \Rightarrow 6 v_{11} = 2 v_{12} \Rightarrow 3 v_{11} = v_{12}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} x + a_{12} y \\ a_{21} x + a_{22} y \end{pmatrix}$$

$$\boxed{v_{12} = 3 v_{11}}$$

$v_{11} = 1$

$v_{12} = 3 v_{11} = 3$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

$$\boxed{\lambda_1 = 3}$$

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}; \quad \lambda_2 = -2;$$

Solve $\underline{\underline{A}}\, \underline{v_2} = \lambda_2 v_2$

$$\begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = -2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$v_{22} = -2 v_{21}$$

$$6 v_{21} + v_{22} = -2 v_{22} \Rightarrow 6 v_{21} = -3 v_{22}$$

$$\boxed{v_{22} = -2 v_{21}}$$

$$v_{21} = 1$$

$$v_{22} = -2 \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix}; \quad \frac{d}{dx}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \underline{\underline{A}}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix}$$

$$\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3x} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2x};$$

Eigenpairs of $\begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \Rightarrow \lambda_1 = 3;\ v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda_2 = -2;\ v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

IF λ_1 and λ_2 are eigenvalues of $\underline{A}$ with eigenvectors v_1 and v_2 then solution of $\frac{d}{dx}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \underline{\underline{A}}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix}$ is

$$\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = c_1 \underline{v}_1 e^{\lambda_1 x} + c_2 \underline{v}_2 e^{\lambda_2 x};$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix};$$

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

$$\dot{x} \equiv \frac{d}{dt}x(t) = a\,x(t) + b\,y(t)$$

$$\dot{y} \equiv \frac{d}{dt}\cancel{x}y(t) = c\,x(t) + d\,y(t)$$

Let λ_1, λ_2 be eigenvalues of $\underline{\underline{A}}$, with corresponding eigenvectors v_1, v_2

Then $\underline{x}(t) = \underline{v}\, e^{\lambda t}$ is a solution to

$$\dot{\underline{x}} = \underline{\underline{A}}\, \underline{x} \,;$$

$$\frac{d}{dt} \underline{x}(t) = \frac{d}{dt}\left(\underline{v}\, e^{\lambda t}\right) = \underline{v} \frac{d}{dt} e^{\lambda t} = \underline{v}\, \lambda e^{\lambda t};$$

$$\underline{\underline{A}} \cdot \underline{x} = \underline{\underline{A}} \cdot \underline{v}\, e^{\lambda t} = e^{\lambda t}\, \underline{\underline{A}}\, \underline{v} = e^{\lambda t} \lambda\, \underline{v} = \frac{d}{dt} \underline{x}$$

$$\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$$

$$\frac{d}{dt}\underline{x}(t) - \underline{\underline{A}}\,\underline{x}(t) =$$

$$= c_1 \underline{v}_1 \lambda_1 e^{\lambda_1 t} + c_2 \underline{v}_2 \lambda_2 e^{\lambda_2 t} -$$

$$- \underline{\underline{A}}\left(c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}\right)$$

$$= c_1 v_1 \lambda_1 e^{\lambda_1 t} + c_2 \underline{v}_2 \lambda_2 e^{\lambda_2 t} - c_1 e^{\lambda_1 t} \underbrace{\underline{\underline{A}}\,\underline{v}_1}_{\lambda_1 \underline{v}_1} - c_2 e^{\lambda_2 t} \underbrace{\underline{\underline{A}}\,\underline{v}_2}_{\lambda_2 v_2} = \varnothing$$

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathrm{Det}\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = \mathrm{Det}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \mathrm{Det}\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} =$$

$$= (\lambda - a)(\lambda - d) - bc =$$

$$= \lambda^2 - (a+d)\lambda - bc = 0$$

Examples

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\mathrm{Det}(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \mathrm{Det}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \mathrm{Det}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = \varnothing$$

$$\lambda_1 = 1, \quad \lambda_2 = 2;$$

- $\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}; \ \lambda_1 = 1;$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \Rightarrow \begin{matrix} v_{11} = v_{11} \\ 2v_{12} = v_{12} \end{matrix}$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $\lambda_2 = 2;$ $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$

$$\begin{aligned} v_{21} &= 2 v_{21} \\ 2 v_{22} &= 2 v_{22} \end{aligned} \Rightarrow \begin{aligned} v_{21} &= \emptyset \\ v_{22} &= 1 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{cases} \nearrow \lambda_1 = 1;\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \searrow \lambda_2 = 2;\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

<u>Example</u>

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\lambda_1 = \lambda_2 = 1;$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 1\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$\left.\begin{matrix} v_{11} = v_{11} \\ v_{12} = v_{12} \end{matrix}\right\}$$

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\mathrm{Det}(A - \lambda I) = \mathrm{Det}\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= (2-\lambda)^2 - 1 = \emptyset = (\lambda - 2)^2 - 1$$

$$(\lambda - 2)^2 = 1$$

$$\lambda - 2 = \pm 1$$

$$\lambda = 2 \pm 1 = 1, 3$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$2a + b = a \Rightarrow \quad a + b = \emptyset \Rightarrow a = -b$$

$$a + 2b = b \Rightarrow \quad a = -b$$

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$