Euler Equations

$$ax^2 y''(x) + bx\,y'(x) + c\,y(x) = 0;$$

$$y(x) = x^r; \qquad ar(r-1) + br + c = 0$$

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}, \quad r_1 \neq r_2$$

$x > 0$,

$$y(x) = (-x)^r; \qquad y(x) = |x|^r$$

<u>Example</u>

$$x^2 y''(x) - x\,y'(x) + y(x) = 0$$

$$r(r-1) - r + 1 = 0$$

$$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0;$$

$$y(x) = C_1 |x| + C_2 |x| \ln|x|$$

IF $r_1 = r_2 = r$

$$y(x) = C_1 x^r + C_2 x^r \cdot \ln x;$$

$$\Downarrow$$

$$y(x) = C_1 |x|^r + C_2 |x|^r \ln |x|$$

Example

$$x^2 y''(x) + 3x y'(x) + 5 y(x) = 0;$$

$$y(x) = x^r;$$

$$r(r-1) + 3r + 5 = 0;$$

$$r^2 + 2r + 5 = 0;$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} =$$

$$= -1 \pm \sqrt{1-5} = -1 \pm 2i$$

$$y(x) = \frac{C_1}{|x|} \cos(2 \ln|x|) + \frac{C_2}{|x|} \sin(2 \ln|x|)$$

Complex roots: $y(x) = C_1 |x|^{\operatorname{Re} r} \cos((\operatorname{Im} r) \ln|x|) + C_2 |x|^{\operatorname{Re} r} \sin((\operatorname{Im} r) \ln|x|)$

Example

$$x^2 y''(x) + x y'(x) - y(x) = \sqrt{x}; \quad x \geq 0;$$

Hom $\quad x^2 y_0'' + x y_0' - y_0 = 0$

$$y(x) = x^r$$

$$r(r-1) + r - 1 = 0$$

$$r^2 = 1, \quad r = \pm 1;$$

$$y_0(x) = \frac{c_1}{x} + c_2 x; \qquad c_1 \leftarrow u_1(x)$$

Variation of a parameter: $\quad c_2 \leftarrow u_2(x)$

$$y(x) = \frac{u_1(x)}{x} + u_2(x)\, x; \qquad y_1(x) = \frac{1}{x}$$

$$y_2(x) = x$$

$$W(x) = \mathrm{Det}\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \mathrm{Det}\begin{pmatrix} 1/x & x \\ -1/x^2 & 1 \end{pmatrix} = 2/x$$

$$u_1(x) = -\int dx \frac{y_2(x)\, f(x)}{W(x)};$$ [redacted]

$$u_2(x) = \int dx \frac{y_1(x)\, f(x)}{W(x)}$$

$f(x)$ is a RHS of nonhomogeneous equation

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = f(x)$$

$$y''(x) + \frac{1}{x} y'(x) - \frac{y(x)}{x^2} = \frac{\sqrt{x}}{x^2} = x^{-3/2}$$

$$u_1(x) = -\int dx \frac{x\, x^{-3/2}}{2/x} = -\int \frac{1}{2} \sqrt{x}\, dx =$$

$$= -\frac{1}{2} \cdot \frac{2}{3} x^{3/2} = -\frac{1}{3} x^{3/2}$$

$$u_2(x) = \int dx \; \frac{1}{x} x^{-3/2} \cdot \frac{1}{2/x} =$$

$$= \frac{1}{2} \int dx \cdot x^{-3/2} = \frac{1}{2}(-2)\, x^{-1/2} =$$

$$= -\frac{1}{\sqrt{x}}$$

$$y(x) = \frac{c_1}{x} + c_2 x + \left(-\frac{1}{3} x^{3/2}\right) \cdot \frac{1}{x} - \frac{1}{\sqrt{x}} \cdot x$$

$$= \underbrace{\frac{c_1}{x} + c_2 x}_{} \underbrace{- \frac{4}{3}\sqrt{x}}_{};$$

Method of undetermined coefficients

$$x^2 y''(x) + x y'(x) - y(x) = \sqrt{x}$$

$$y(x) = A\sqrt{x};$$

$$y'(x) = \frac{1}{2} A \frac{1}{\sqrt{x}} = \frac{A}{2} x^{-1/2}$$

$$y''(x) = -\frac{A}{4} \frac{1}{x^{3/2}} = -\frac{A}{4} x^{-3/2}$$

$$-\frac{A}{4} x^2 x^{-3/2} + \frac{A}{2} x\, x^{-1/2} - A\sqrt{x} = \sqrt{x}$$

$$-\frac{A}{4} + \frac{A}{2} - A = 1; \quad -\frac{3}{4} A = 1 \Rightarrow A = -\frac{4}{3}$$

$$y(x) = \frac{C_1}{x} + C_2 x - \frac{4}{3}\sqrt{x}$$

Review

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x) = 0$$

$$y(x) = C_1\,y_1(x) + C_2\,y_2(x);$$

$$C_1 y_1 + C_2 y_2 = 0 \iff C_1 = C_2 = 0$$

Linear indepence

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \neq 0$$

Constant coefficients

$$a y''(x) + b y'(x) + c\, y(x) = \emptyset;$$

$$y(x) = e^{\lambda x} = \exp(\lambda x)$$

$$a\lambda^2 + b\lambda + c = \emptyset;$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 - 4ac > \emptyset$: $y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
- $b^2 - 4ac = \emptyset$: $y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$
- $b^2 - 4ac < \emptyset$

IF $b^2 - 4ac < 0$

$$\lambda_{1,2} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

$$= \underbrace{-\frac{b}{2a}}_{\gamma} \pm i \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\omega} = \gamma \pm i\omega$$

$$y(x) = C_1 e^{\gamma x + i\omega x} + C_2 e^{\gamma x - i\omega x}$$

$$\left.\begin{aligned} &= D_1 e^{\gamma x} \cos \omega x + D_2 e^{\gamma x} \sin \omega x \\ &= R e^{\gamma x} \cos(\omega x - \varphi) \\ &= R e^{\gamma x} \sin(\omega x + \psi) \end{aligned}\right] \text{any}$$

UNDETERMINED COEFFICIENTS

$$a\,y''(x) + b\,y'(x) + c\,y(x) = f(x);$$

$$\cancel{f(x) = a_1 + b_1 x + c_1 x^2 + \dots +}$$

- $f(x) = a_0 + a_1 x + a_2 x^2 + \dots +$ [redacted] $a_n x^n;$ $\quad \Big\| \quad y_p(x) = b_0 + b_1 x + \dots + b_n x^n$

- $f(x) = A \sin Bx$

  or

  $f(x) = C \cos Bx$ $\qquad y_p = D \sin Bx + E \cos Bx$

  $f(x) = a e^{bx}$ $\qquad y_p(x) = c e^{bx};$

$$f(x) = \sum_{k=0}^{n} a_k x^k \cdot e^{\alpha x} \cdot \cos \beta x$$

or

$$f(x) = \sum_{k=0}^{n} a_k x^k e^{\alpha x} \sin \beta x$$

$$y_p(x) = \sum_{k=0}^{n} A_k x^k \cdot e^{\alpha x} (B \sin \beta x) \cancel{+ C \cos \beta x)}$$

$$+ \sum_{k=0}^{n} C_k x^k e^{\alpha x} C \sin \beta x$$

IF any part of a particular solution is a solution to homogeneous equation

$$y_p \to y_p \cdot x \to y_p \cdot x^2$$

$$m\ddot{x} + 2\gamma\dot{x} + kx = F\cos\omega t$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f\cos\omega t;$$

- free oscillator $\beta = f = 0;$

$$x = C_1\cos\omega_0 t + C_2\sin\omega_0 t$$

$$= R\cos(\omega_0 t - \varphi) = R\sin(\omega_0 t + \psi)$$

$$T = \frac{2\pi}{\omega_0}$$

$f = \emptyset, \ \beta > \emptyset$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \emptyset$$

$$x = e^{\lambda x}$$

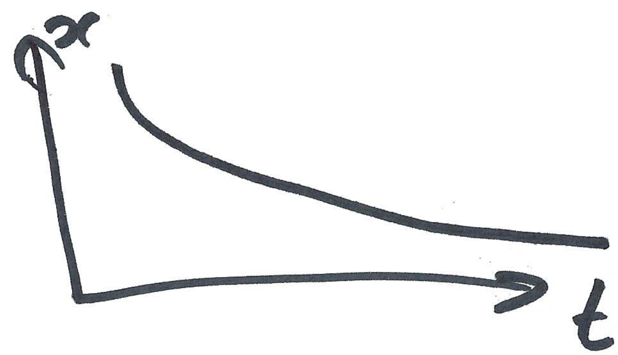
$$\lambda^2 + 2\beta\lambda + \omega_0^2 = \emptyset$$

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

→ overdamped: $\beta^2 > \omega_0^2$

$$x(t) = c_1 e^{\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + c_2 e^{\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}$$

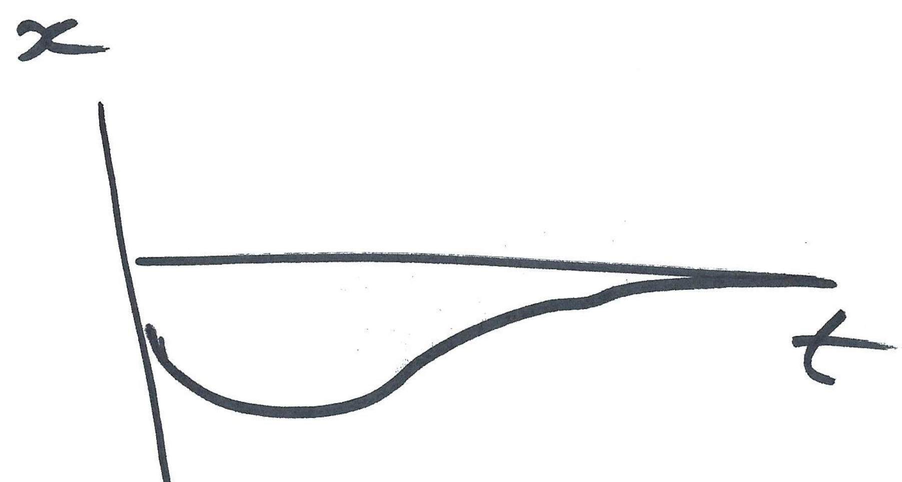
$$\left(-\beta \pm \sqrt{\beta^2 - \omega_0^2}\right) < \emptyset$$



- $\beta^2 = \omega_0^2$ - critically damped:

$$\lambda_{1,2} = -\beta;$$

$$y(t) = C_1 e^{-\beta t} + C_2 \cdot t \cdot e^{-\beta t}$$



underdamped

$$\beta^2 < \omega_0^2$$

$$\lambda_{1,2} = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

$$y(x) = C_1 e^{-\beta t} \cos\left(\sqrt{\omega_0^2 - \beta^2}\, t\right)$$

$$+ C_2 e^{-\beta t} \sin\left(\sqrt{\omega_0^2 - \beta^2}\, t\right)$$

$$= R e^{-\beta t} \cos\left(\sqrt{\omega_0^2 - \beta^2}\, t\right)$$

$$= R e^{-\beta t} \sin\left(\sqrt{\omega_0^2 - \beta^2}\, t\right)$$

$$\tilde{T} = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}$$ quasi period

Forced

$\gamma = \emptyset, \quad f \neq \emptyset$

resonant forcing

nonresonant forcing

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\frac{d}{dt}x(t) = a\,x(t) + b\,y(t)$$

$$\frac{d}{dt}y(t) = c\,x(t) + d\,y(t)$$

a, b, c, d are given

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}$$