$$5\,\mathrm{kg}\,\ddot{x}(t) + \frac{50\,\mathrm{kg}}{\mathrm{sec}}\dot{x} + \frac{500\,\mathrm{kg}}{\mathrm{sec}^2}x = 10\sin\left(\frac{t}{2\,\mathrm{sec}}\right)\mathrm{N}$$

$$x(t) = y(\tau)\cdot 1\,\mathrm{meter}$$

$$t = 1\,\mathrm{sec}\cdot\tau; \quad [\tau] = [y] = 1$$

dimensions

$$\frac{d}{dt} = \frac{1}{1\,\mathrm{sec}}\frac{d}{d\tau}; \quad \frac{d^2}{dt^2} = \frac{1}{\mathrm{sec}^2}\frac{d^2}{d\tau^2}$$

$$5\,\frac{\mathrm{kg}\cdot\mathrm{meter}}{\mathrm{sec}^2}\frac{d^2}{d\tau^2}y(\tau) + 50\,\frac{\mathrm{kg}\cdot\mathrm{meter}}{\mathrm{sec}^2}\frac{d}{d\tau}y(\tau) + 500\,\frac{\mathrm{kg}\cdot\mathrm{meter}}{\mathrm{sec}^2}y(\tau) = 10\sin\left(\frac{\tau}{2}\right)\mathrm{N}$$

$$\ddot{u} + 2\dot{u} + 2u = 5\sin t \qquad 1$$

$$u(\emptyset) = \dot{u}(\emptyset) = \emptyset; \qquad 2$$

Homogeneous: $\ddot{u}_0 + 2\dot{u} + 2u = \emptyset,$ 3

$$u_0 = e^{\lambda t} \qquad 4$$

$$\lambda^2 + 2\lambda + 2 = \emptyset, \qquad 5$$

$$\cancel{\lambda_{12} = -\lambda +}$$

$$\lambda_{12} = -1 \pm \sqrt{1^2 - 2} = \qquad 6$$

$$= -1 \pm i\sqrt{1}$$

$$u_0(t) = A e^{-t} \sin \sqrt{1}\, t$$

$$+ B e^{-t} \cos \sqrt{1}\, t$$

$$x^2 + 2bx + \cancel{4}c = 0$$

$$x_{12} = \frac{-2b \pm \sqrt{4b^2 - 4c}}{2} =$$

$$= -b \pm \sqrt{b^2 - c}$$

$$x^2 + 2x + 2 = 0$$

$$x_{12} = \frac{-2 \pm \sqrt{4 - 8}}{2}$$

$$= -1 \pm i$$

$$\ddot{u} + 2\dot{u} + 2u = 5\sin t$$

$$u(t) = A\sin t + B\cos t$$

$$\dot{u}(t) = A\cos t - B\sin t$$

$$\ddot{u}(t) = -A\sin t - B\cos t$$

$$\overbrace{-A\sin t - B\cos t}^{\ddot{u}} + \overbrace{2A\cos t - 2B\sin t}^{2\dot{u}} + \overbrace{2A\sin t + 2B\cos t}^{2u} = 5\sin t$$

$$(\sin t)(-A - 2B + 2A - 5) + (\cos t)(-B + 2A + 2B) = 0$$

$$\begin{cases} A - 2B = 5 \\ 2A + B = 0, \end{cases}$$

$$B = -2A$$

$$A + 4A = 5 \Rightarrow A = 1,\ B = -2$$

$$u(t) = Ae^{-t}\sin t + Be^{-t}\cos t + \sin t - 2\cos t$$

$$\dot{u}(t) = -Ae^{-t}\sin t + Ae^{-t}\cos t$$
$$- Be^{-t}\cos t - Be^{-t}\sin t + \cos t + 2\sin t$$

$u(t=0) = B - 2 = 0, \quad B = 2$

$\dot{u}(t=0) = A - B + 1 = 0, \quad A = B - 1 = 1$

$$u(t) = e^{-t}\sin t + 2e^{-t}\cos t + \sin t - 2\cos t$$

$$\lim_{t \to \infty} u(t) = \sin t - 2\cos t$$

Euler Equation

$a x^2 y''(x) + b x y'(x) + c\, y(x) = \emptyset;$

Homogeneous ($y = \emptyset$ is a solution)

Linear, second order ($y''(x)$)

Does NOT have constant coefficients

$y(x) = x^{\lambda}, \; y'(x) = \lambda x^{\lambda-1}; \; y''(x) = \lambda(\lambda-1)x^{\lambda-2}$

$$(a\lambda(\lambda-1) + b\lambda + c) \cdot x^{\lambda} = \emptyset$$

$$a\lambda^2 + (b-a)\lambda + c = \emptyset;$$

$$\lambda_{12} = \frac{a - b \pm \sqrt{(b-a)^2 - 4c}}{2a}$$

- 2 real distinct roots $\lambda_1 \neq \lambda_2$, $Im(\lambda_1) = Im(\lambda_2) = \emptyset$
- 1 real double root $\lambda_1 = \lambda_2$; $Im(\lambda_1) = \emptyset$
- 2 complex roots that are cc (complex conjugate) of each other

2 real roots : $\lambda_1 \neq \lambda_2$

$y_1(x) = x^{\lambda_1}$; $y_2(x) = x^{\lambda_2}$;

$$W(x) = Det\begin{pmatrix} x^{\lambda_1} & x^{\lambda_2} \\ \lambda_1 x^{\lambda_1 - 1} & \lambda_2 x^{\lambda_2 - 1} \end{pmatrix} = (\lambda_2 - \lambda_1)\, x^{\lambda_1 + \lambda_2 - 1}$$

$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$;

- $2x^2y'' + 3xy' - y = 0;$

$y = x^{\lambda_1};$

$$2\lambda_1(\lambda_1 - 1) + 3\lambda - 1 = 0$$

$$2\lambda_1^2 + \lambda - 1 = 0;$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1,\ 1/2$$

$$y(x) = C_1/x + C_2\sqrt{x}\ ,\quad x > 0$$

$$y(x) = \frac{C_1}{|x|} + C_2\sqrt{|x|}$$

2 real roots that coincide

$$x^2 y'' + (2\beta + 1) x y' + \beta^2 y = 0;$$

$y(x) = x^\lambda$; $\quad \lambda \cdot (\lambda - 1) + (2\beta + 1)\lambda + \beta^2 = 0$

$$\lambda^2 + 2\beta\lambda + \beta^2 = 0$$

$$(\lambda + \beta)^2 = 0;$$

$\lambda = -\beta$; $\quad y_1(x) = \dfrac{1}{x^\beta} = 1/x^\beta$

$$\cancel{x^2 \frac{d^2}{dx^2} + x(2\beta + 1)\frac{d}{dx}}$$

$$\left(x^2 \frac{d^2}{dx^2} + (2\beta + 1) x \frac{d}{dx} + \beta^2\right) y(x) = 0$$

$$x^2\frac{d^2}{dx^2} + x(2\beta+1)\frac{d}{dx} + \beta^2 \overset{?}{=} \checkmark$$

$$\overset{?}{=} \left(x\frac{d}{dx} + \beta\right)^2 =$$

$$= \left(x\frac{d}{dx} + \beta\right)\left(x\frac{d}{dx} + \beta\right) =$$

$$= x\frac{d}{dx} + x^2\frac{d^2}{dx^2} + 2\beta x\frac{d}{dx} + \beta^2$$

$$\left(x\frac{d}{dx} + \beta\right)\underbrace{\left(x\frac{d}{dx} + \beta\right)y(x)}_{z(x)} = \emptyset \qquad \boxed{z = x^{-\beta}}$$

$$xz' + \beta z = \emptyset$$

$$x\frac{dz}{dx} = -\beta z \Rightarrow \underbrace{\int\frac{dz}{z}}_{=\ln z} = -\int\beta\frac{dx}{x} = -\beta\ln x$$

$$= \ln x^{-\beta}$$

$$x y' + \beta y = x^{-\beta}$$

$$y' + \frac{\beta}{x} y = x^{-\beta+1}$$

$$\mu(x) = e^{\int \beta/x \, dx} = e^{\beta \int \frac{dx}{x}} = e^{\beta \ln x} =$$

$$= e^{\ln x^{\beta}} = x^{\beta}$$

$$\cancel{x^{\beta+1}} \left(x^{\beta} y' + \beta x^{\beta-1} \right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(x^{\beta} y(x) \right) = \frac{1}{x}$$

$$x^{\beta} y(x) = \ln x$$

$$y(x) = x^{-\beta} \ln x$$

$$y(x) = C_1 x^\lambda + C_2 (\ln x) \cdot x^\lambda$$

$$y(x) = C_1 |x|^\lambda + C_2 \ln|x| \cdot |x|^\lambda$$

2 roots that are complex conjugate of each other

$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$$

$$\boxed{x = e^{\ln x}}$$

$$\ln x^a = a \ln x$$

$\lambda_1 = a + i b,$ $\quad a = \text{Re}\,\lambda_1 = \text{Re}\,\lambda_2$

$\lambda_2 = a - i b$ $\quad b = \text{Im}\,\lambda_1 = -\text{Im}\,\lambda_2$

$$y(x) = C_1 x^{a+ib} + C_2 x^{a-ib}$$

$$= C_1 \exp(\ln(x^{a+ib})) + C_2 \exp(\ln(x^{a-ib}))$$

$$= C_1 \exp((a+ib)\ln x) + C_2 \exp((a-ib)\ln x)$$

$$= C_1 \exp(a \ln x) \exp(i b \ln x)$$

$$+ C_2 \exp(a \ln x) \exp(-i b \ln x)$$

$$y(x) = C_1 \exp(\ln x^a) \cos(b \ln x) +$$
$$+ i C_1 \exp(\ln X^a) \sin(b \ln x)$$
$$+ C_2 \exp(\ln X^a) \cos(b \ln x)$$
$$- i C_2 \exp(\ln X^a) \sin(b \ln x)$$
$$= \overbrace{\boxed{(C_1 + C_2)}}^{D_1} X^a \cos(b \ln X)$$
$$+ \underbrace{\boxed{i(C_1 - C_2)}}_{D_2} X^a \sin(b \ln x)$$
$$y(x) = D_1 X^{\mathrm{Re}\,\lambda} \cos((\mathrm{Im}\,\lambda) \ln x)$$
$$+ D_2 X^{\mathrm{Re}\,\lambda} \sin((\mathrm{Im}\,\lambda) \ln x)$$

$$x^2 y''(x) + x y'(x) + y(x) = \emptyset$$

$$\lambda(\lambda - 1) + \lambda + 1 = \emptyset$$

$$\lambda^2 + 1 = \emptyset$$

$$\lambda = \pm i$$

$$y(x) = D_1 \cos(\ln(x)) + D_2 \sin(\ln(x))$$