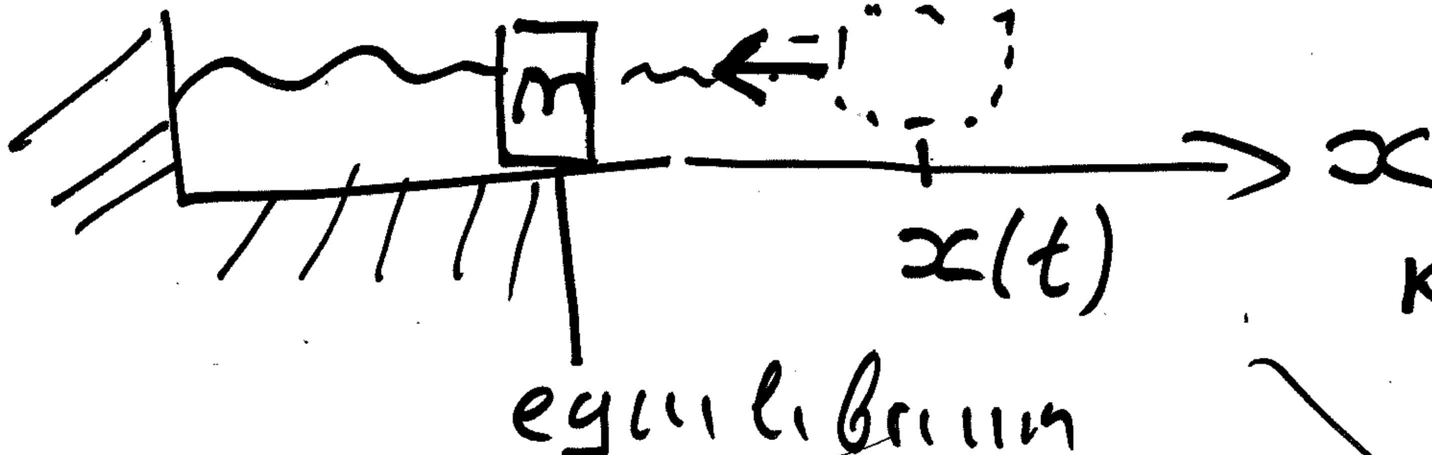


$\kappa \equiv k$

$$\bar{F} = -k\bar{x} \leftarrow$$

$$\cancel{F = -\bar{\beta}x + \frac{\alpha}{2}x^2 + \frac{\beta x^3}{3}}$$

$$\bar{F} = m\bar{a} = $$

$$= m\ddot{\bar{x}}$$

$$= -\kappa\bar{x}$$

$$\ddot{\bar{x}}(t) = \frac{d^2}{dt^2}\underline{x}(t)$$

$$m\ddot{x} = -kx \Rightarrow m\ddot{x} + kx = \emptyset$$

$$x = x(t)$$

$\ddot{x} + \frac{k}{m} x = \emptyset;$ $\omega_0^2 = \frac{k}{m}$

$\ddot{x} + \omega_0^2 x = \emptyset;$

+ friction $\underbrace{+\ 2\beta\dot{x}}$ LHS

+ external force RHS $= f \cos \omega t$

$d = 4$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f \cos \omega t$$

$\bar{x}$ - vector - $\underline{x}$ - d components

$d = 3$ we

$d = 2$ paper

$d = 1$ straight line

external force : $f(t)$

Fourier transform

$$f(t) = \sum_{n=1}^{\infty} A_n \cos(nt)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f\cos(\omega t);$$

- Free oscillator $\beta = f = 0;$

$$\ddot{x} + \omega_0^2 x = 0;$$

$$x = e^{\lambda t} \Rightarrow \lambda^2 + \omega_0^2 = 0$$

$$\lambda = \pm i\omega_0$$

- $x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t};$
- $x(t) = D_1\cos\omega_0 t + D_2\sin\omega_0 t;$
- $x(t) = R\cos(\omega_0 t - \varphi);$
- $x(t) = R\cdot\sin(\omega_0 t + \Psi);$

$$x(t) = \underbrace{R}_{\text{amplitude}} \cos(\omega_0 t - \underbrace{\varphi}_{\text{phase}});$$

$$x(t) = x(t+T)$$

$$R\cos(\omega_0 t - \varphi) = R\cos(\omega_0 t + \omega_0 T - \varphi)$$

$$\omega_0 T = 2\pi$$

$$T = \frac{2\pi}{\omega_0} \text{ - period}$$

Add friction

"damped oscillator"

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x(t) = e^{\lambda t}$$

$$\dot{x}(t) = \lambda e^{\lambda t}$$

$$\ddot{x}(t) = \lambda^2 e^{\lambda t}$$

"usually" $\beta \geq 0$

$$\ddot{\lambda} + 2\beta\lambda + \omega_0^2 = 0$$

$$\lambda_{1,2} = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$\rightarrow \beta^2 > \omega_0^2$

$\rightarrow \beta^2 = \omega_0^2$

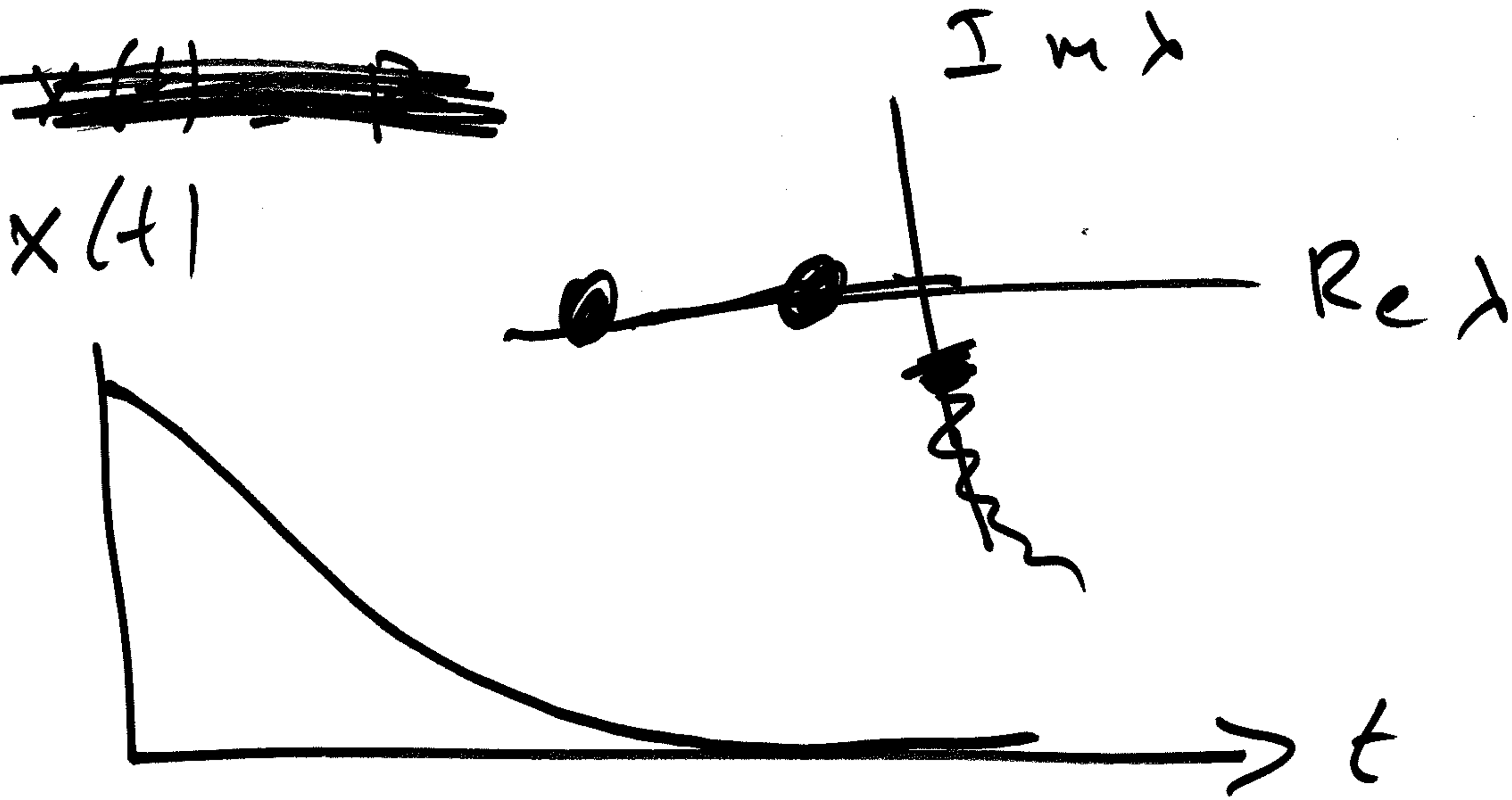
$\rightarrow \beta^2 < \omega_0^2$

$\beta^2 > \omega_0^2$ overdamped oscillator

$\lambda_{1,2}$ are two real roots

$0 \leq \sqrt{\beta^2 - \omega_0^2} < \beta$ both are negative overdamped oscillator

- $x(t) = C_1 e^{\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}$



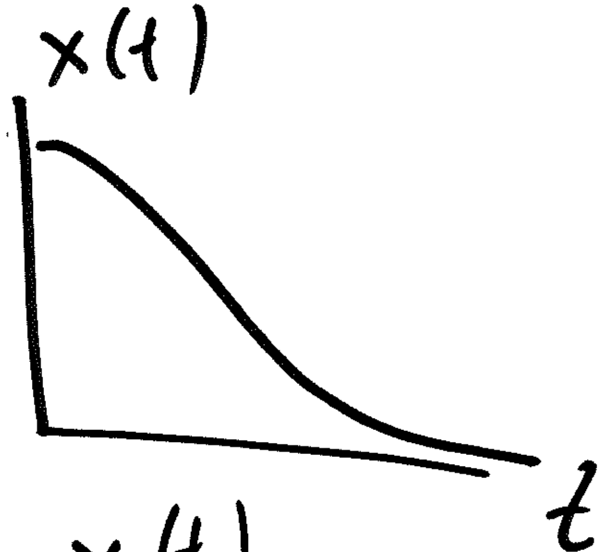
$$\beta^2 = \omega_0^2$$

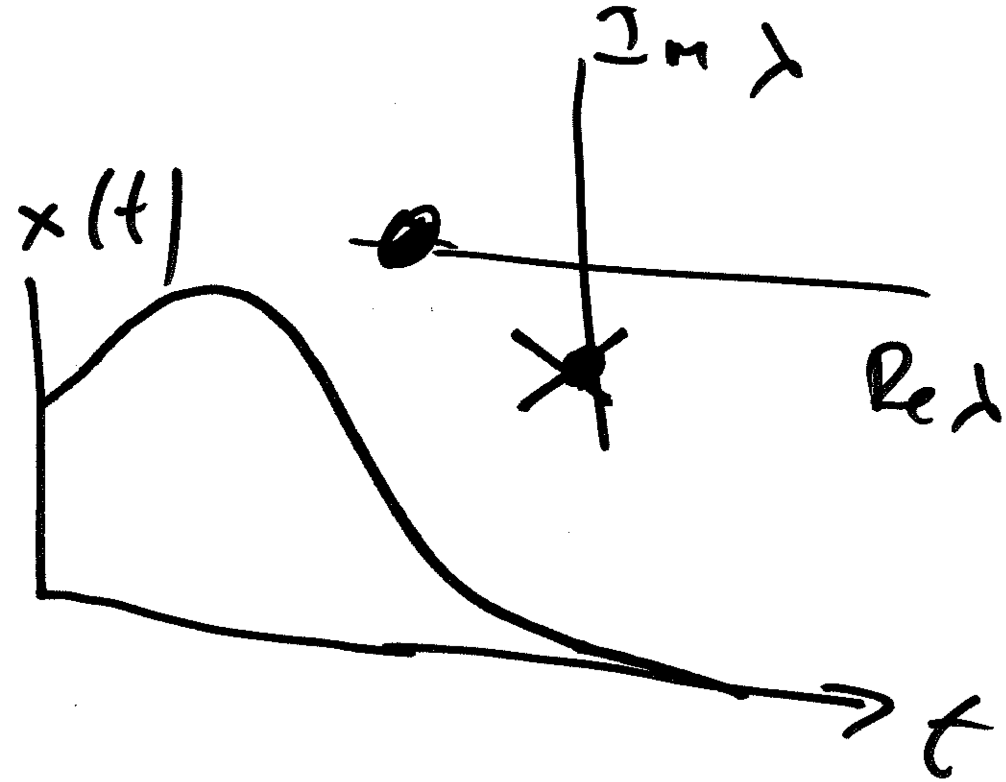
critically damped oscillator

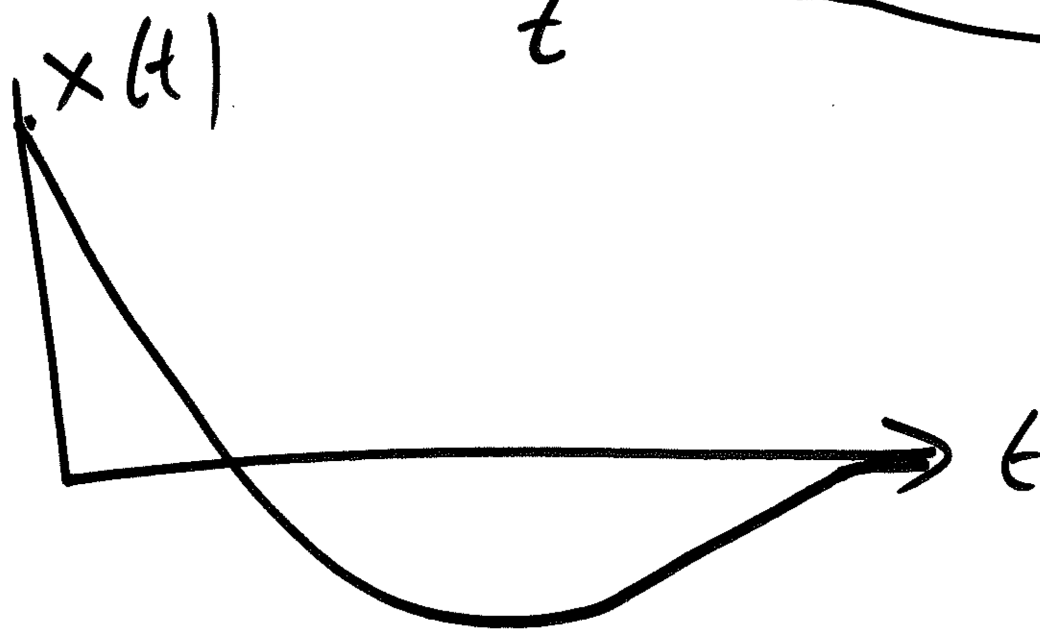
$$\lambda_1 = \lambda_2 = -\beta$$

$$x(t) = C_1 e^{-\beta t} + C_2 \cdot t \cdot e^{-\beta t}$$

Scenarios:





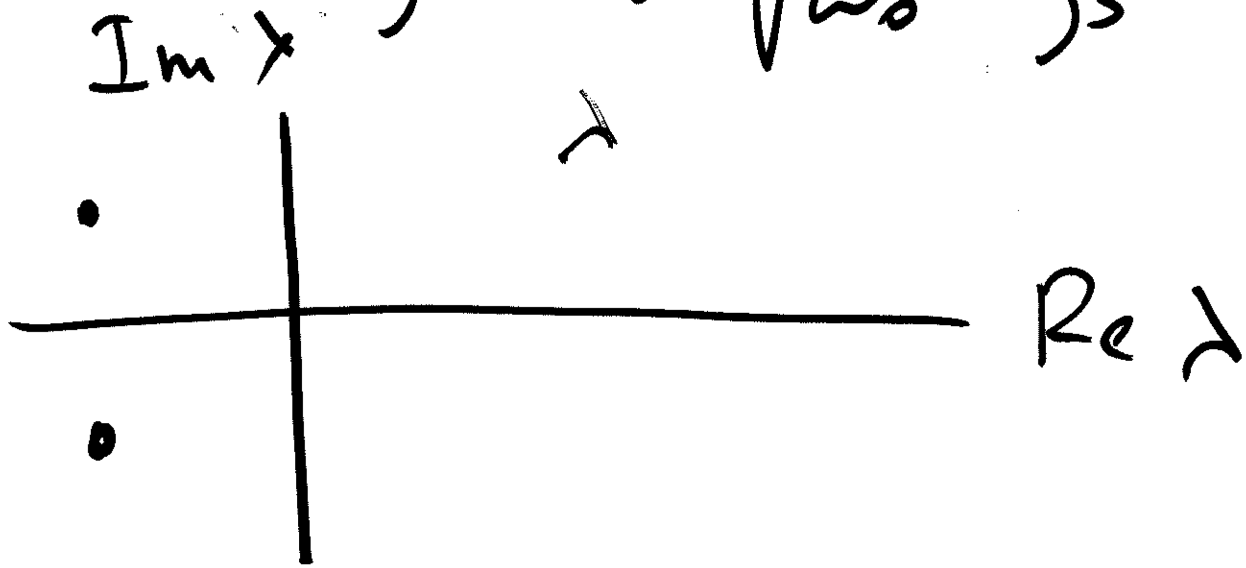


$$\beta^2 < \omega_0^2$$

underdamped oscillator

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$= -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

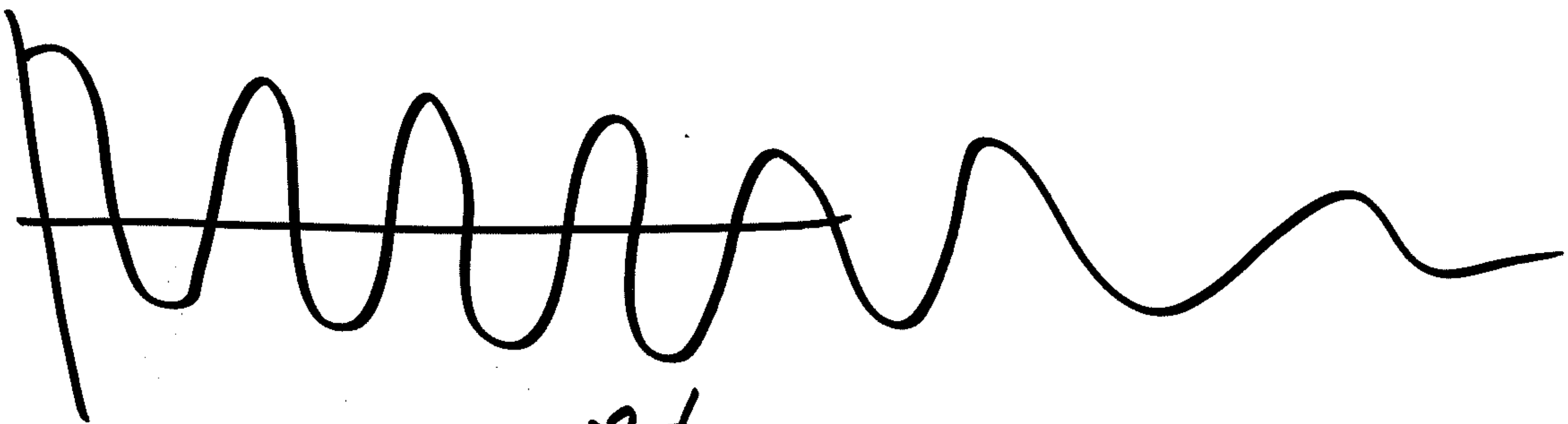


$$x(t) = C_1 e^{-\beta t} \cos\left(\sqrt{\omega_0^2 - \beta^2}\, t\right) + C_2 e^{-\beta t} \sin\left(\sqrt{\omega_0^2 - \beta^2}\, t\right)$$

$$x(t) = R e^{-\beta t} \cos\left(\sqrt{\omega_0^2 - \beta^2}\, t - \varphi\right)$$

$$x(t) = R e^{-\beta t} \sin\left(\sqrt{\omega_0^2 - \beta^2}\, t + \varphi\right)$$

Small β



$$x(t) = \underbrace{R e^{-\beta t}}_{\text{Amplitude}} \cos\left(\sqrt{\omega_0^2 - \beta^2}\, t + \varphi\right)$$

- "Envelope function"

$$\sqrt{\omega_0^2 - \beta^2} \leq \omega_0 \quad \text{if } \beta \geq 0$$

Quasi perion

$$x(t+T) \neq x(t)$$

for any T

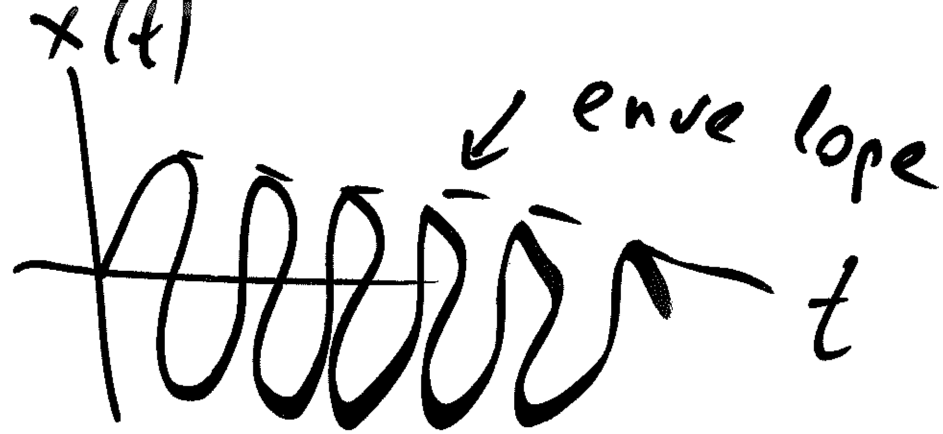
If $\emptyset < \beta << \omega_0$

$$x(t+T) \approx x(t)$$

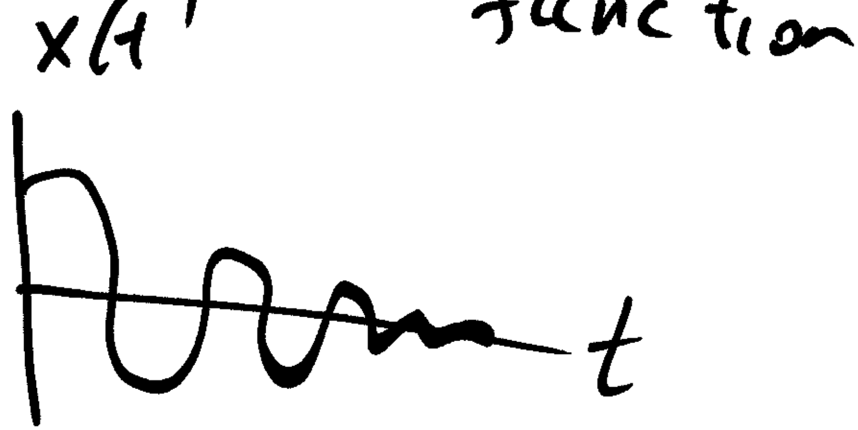
$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}$$

$$\lim_{\beta \to \emptyset} \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\omega_0}$$

small β



Larger β.



Big β

Mathematica

Matlab

Mapple

Forced and damped harmonic oscillators

Forced oscillator

$\beta = \emptyset, \; f \neq \emptyset$

$$\ddot{x}(t) + \omega_0^2 x(t) = f \cos \omega t;$$

$\omega \neq \omega_0$ nonresonant forcing

$\omega = \omega_0$ resonant forcing

— $\omega \neq \omega_0$;

Homogeneous equation $\ddot{x}_0 + \omega_0^2 x_0 = \emptyset$.

$$\ddot{x} + \omega_0^2 x = f \cos \omega t$$

→ method of undetermined coefficents

→ variation of a parameter

$$x_p(t) = A \cos(\omega t)$$

A is a constant to be found.

$$\dot{x}_p(t) = -A\omega \sin(\omega t)$$

$$\ddot{x}_p(t) = -A\omega^2 \cos(\omega t)$$

$$-A\omega^2 \cos\omega t + \omega_0^2 A \cos\omega t = f\cos\omega t;$$

$\cos\omega t \neq 0$ if $\omega t \neq \frac{\pi}{2} + \pi n$

$$A(\omega_0^2 - \omega^2) = f$$

$$A = \frac{f}{\omega_0^2 - \omega^2}$$

$$x(t) = R\cos(\omega_0 t - \varphi) + \frac{f}{\omega_0^2 - \omega^2}\cos(\omega t)$$

$$x(t) = \underbrace{R\sin(\omega_0 t + \psi)}_{\text{free oscillation}} + \underbrace{\frac{f}{\omega_0^2 - \omega^2}\cos(\omega t)}_{\text{forced oscillation}}$$