Variation of Parameter — 1

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x); \quad (*)$$

Homogeneous: $y_0''(x) + p(x)y_0'(x) + q(x)y_0(x) = 0. \quad (**)$

$$y_0(x) = C_1 y_1(x) + C_2 y_2(x);$$

$y_1(x)$ and $y_2(x)$ are solutions to $(**)$;

Replace

$$C_1 \leftarrow u_1(x)$$

$$C_2 \leftarrow u_2(x)$$

Look for particular solution of $(*)$

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x);$$

$$y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x);$$

zero

$$y'(x) = u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) +$$

$$+ \; u_1(x)\, y_1'(x) + u_2(x)\, y_2'(x);$$

Choose $\quad u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0$

$$\cancel{y'(x) = u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x);}$$

$$y'(x) = u_1(x)\, y_1'(x) + u_2(x)\, y_2'(x);$$

$$y''(x) = u_1'(x)\, y_1'(x) + u_2'(x)\, y_2'(x) +$$

$$+ \; u_1(x)\, y_1''(x) + u_2(x)\, y_2''(x);$$

$$u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) + p(x)\Big(\underbrace{u_1(x)y_1'(x)}_{c} + \underbrace{u_2(x)y_2'(x)}_{e}\Big) + g(x)\Big(\underbrace{u_1(x)y_1(x)}_{b} + \underbrace{u_2(x)y_2(x)}_{f}\Big) + \underbrace{u_1(x)\,y_1''(x)}_{a} + \underbrace{u_2(x)\,y_2''(x)}_{d} = f(x);$$

$$u_1'(x)\,y_1'(x) + u_2'(x)y_2'(x) + u_1(x)\Big(\overset{a}{y_1''(x)} + \overset{c}{p(x)y_1'(x)} + \overset{b}{g(x)y_1(x)}\Big) + u_2(x)\Big(\overset{d}{y_2''(x)} + \overset{e}{p(x)y_2'(x)} + \underset{f}{g(x)y_2(x)}\Big)$$

$$= f(x);$$

$$\left.\begin{aligned} u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) &= f(x) \\ u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) &= \emptyset \end{aligned}\right\}$$

$$u_2'(x) = -\frac{u_1'(x)\,y_1(x)}{y_2(x)}$$

$$\left. u_1'(x)\,y_1'(x) - \frac{u_1'(x)\,y_1(x)\;y_2'(x)}{y_2(x)} = f(x) \right] \cdot y_2(x)$$

$$u_1'(x)\Big(y_2(x)\,y_1'(x) - y_1(x)\,y_2'(x)\Big) = f(x)\,y_2(x)$$

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1(x)\,y_2'(x) - y_2(x)\,y_1'(x)$$

$$u_1'(x) = -\frac{f(x)}{W(x)}\,y_2(x)$$

$$\boxed{u_1(x) = -\int \frac{f(x)\,y_2(x)}{W(x)}\,dx \;;\quad u_2(x) = \int \frac{f(x)\,y_1(x)\,dx}{W(x)}}$$

Ex $y''(x) - 4y'(x) + 3y(x) = -e^{-2x};$

Homogeneous: $y_0'' - 4y_0' + 3y_0 = \emptyset;$

$r^2 - 4r + 3 = \emptyset$

$$r_{12} = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} = 2 \pm 1 = 1, 3;$$

$y_0(x) = C_1 e^{x} + C_2 e^{3x};$

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$$

$$= e^{x} \cdot 3e^{3x} - e^{3x} e^{x} = 2e^{4x}$$

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x);$$

$$y_1(x) \longrightarrow C_1\, y_1(x);$$

$$y_2(x) \longrightarrow C_2\, y_2(x);$$

$$W = \operatorname{Det}\begin{bmatrix} C_1 y_1 & C_2 y_2 \\ C_1 y_1' & C_2 y_2' \end{bmatrix} = C_1 C_2 (y_1 y_2' - y_2 y_1')$$

$$u_1(x) = -\int \frac{f(x)\, C_2\, y_2(x)}{C_1 C_2 (y_1 y_2' - y_2 y_1')} \Rightarrow \frac{1}{C_1}$$

$$y_p(x) = u_1(x)\, C_1\, y_1(x) + u_2(x)\, C_2\, y_2(x)$$

$$y_1(x) = e^{x}; \quad y_2 = e^{3x}$$

$$W(x) = 2e^{4x}$$

$$u_1(x) = -\int \frac{(-)e^{-2x}\, e^{3x}}{2e^{4x}}\,dx = +\frac{1}{2}\int e^{-3x}\,dx =$$

$$= -\frac{1}{6}e^{-3x} = -\frac{e^{-3x}}{6}$$

$$u_2(x) = \int \frac{-e^{-2x}\, e^{x}}{2e^{4x}}\,dx = -\frac{1}{2}\int e^{-5x}\,dx =$$

$$= \frac{1}{10}e^{-5x}$$

$$y_p(x) = -\frac{1}{6}e^{-3x}\cdot e^{x} + \frac{1}{10}e^{-5x}\, e^{3x} = -\frac{1}{15}e^{-2x}$$

$$\overset{\backslash 3}{\frac{1}{10}} - \overset{\backslash 5}{\frac{1}{6}} = \frac{-2}{30} = -\frac{1}{15}$$

$$y'' - 4y' + 3y = -e^{-2x}$$

$$y_r = Ae^{-2x}$$

$$4Ae^{-2x} - A4 \cdot (-2)e^{-2x} + 3Ae^{-2x}$$

$$= -e^{-2x}$$

$$15Ae^{-2x} = -e^{-2x}$$

$$A = -\frac{1}{15}$$

$$y(x) = C_1 e^{x} + C_2 e^{3x} - \frac{1}{15} e^{-2x}$$

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$$y''(x) + y(x) = \mathrm{Tan}(x);$$

$$y_1(x) = \cos x; \quad y_2(x) = \sin(x);$$

$$W = y_1 y_2' - y_1' y_2 = \cos x \cos x + \sin x \cdot \sin x = 1$$

$$u_1(x) = -\int \frac{f(x)\, y_2(x)\, dx}{W(x)} =$$

$$= -\int \frac{\mathrm{Tan}(x) \cdot \sin x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x} dx = -\int \frac{dx}{\cos x} + \int \cos x\, dx$$

$$\int \frac{dx}{\cos x} = + \mathrm{Log}\left[\mathrm{Tan}\, x + \frac{1}{\cos x}\right] + \cancel{\sin x};$$

+ here, no this term

$$u_1(x) = -\mathrm{Log}\left(\mathrm{Tan}\,x + \frac{1}{\cos x}\right) + \sin x + \cancel{\sin x}$$

$$= -\mathrm{Log}\left(\mathrm{Tan}\,x + \frac{1}{\cos x}\right) + \cancel{2}\sin x$$

$$u_2(x) = \int \frac{f(x)\,y_1(x)}{W(x)}\,dx = \int \frac{\tan(x)\cos x}{1}\,dx$$

$$= \int \sin x\,dx = -\cos x$$

$$y(x) = C_1\cos x + C_2\sin x +$$

$$-\left(\mathrm{Log}\left(\tan x + \frac{1}{\cos x}\right)\cdot\cos x + \cancel{2}\sin x\cos x\right.$$

$$-\cos(x)\sin(x) =$$

$$= C_1\cos x + C_2\sin x - \mathrm{Log}\left(\tan x + \frac{1}{\cos x}\right)\cos x + \cancel{\sin x\cos x}$$

$$y''(x) + y(x) = \tan x;$$

$$y''_0(x) + y_0(x) = \emptyset;$$

$$y_0(x) = e^{rx}; \quad r^2+1=\emptyset; \quad r = \pm i$$

$$y(x) = C_1 e^{ix} + C_2 e^{-ix}$$

$$= C_1 \cos x + i C_1 \sin x + C_2 \cos x - i C_2 \sin x$$

$$= (\cos(x))(C_1 + C_2) + i(C_1 - C_2)\sin x$$

$$= D_1 \cos x + D_2 \sin x$$