# Variation of Parameter — 1

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x); \quad (*)$$

Homogeneous: $y_0''(x) + p(x)\, y_0'(x) + q(x)\, y_0(x) = 0. \quad (**)$

$$y_0(x) = C_1\, y_1(x) + C_2\, y_2(x);$$

$y_1(x)$ and $y_2(x)$ are solutions to $(**)$;

Replace

$$C_1 \leftarrow u_1(x)$$

$$C_2 \leftarrow u_2(x)$$

Look for particular solution of $(*)$

$$y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x);$$

$$y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x);$$

zero

$$y'(x) = u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) + u_1(x)\, y_1'(x) + u_2(x)\, y_2'(x);$$

Choose $\quad u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0$

$$\cancel{y'(x) = u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x);}$$

$$y'(x) = u_1(x)\, y_1'(x) + u_2(x)\, y_2'(x);$$

$$y''(x) = u_1'(x)\, y_1'(x) + u_2'(x)\, y_2'(x) + u_1(x)\, y_1''(x) + u_2(x)\, y_2''(x);$$

$$u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) + p(x)\Big(\overset{c}{\underline{u_1(x)y_1'(x)}} + \overset{e}{\underline{u_2(x)y_2'(x)}}\Big) + g(x)\Big(\overset{b}{\underline{u_1(x)y_1(x)}} + \underset{f}{\underline{u_2(x)y_2(x)}}\Big) = f(x);$$

$$+\, \underset{a}{\underline{u_1(x)\,y_1''(x)}} + \overset{d}{\underline{u_2(x)\,y_2''(x)}}$$

$$u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) + u_1(x)\Big(\overset{a}{y_1''(x)} + \overset{c}{p(x)y_1'(x)} + \overset{b}{g(x)y_1(x)}\Big)$$

$$+\, u_2(x)\Big(\overset{d}{y_2''(x)} + \overset{e}{p(x)y_2'(x)} + \underset{f}{g(x)y_2(x)}\Big)$$

$$= f(x);$$

$$\left.\begin{aligned} u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) &= f(x) \\ u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) &= 0 \end{aligned}\right\}$$

$$u_2'(x) = -\frac{u_1'(x)\,y_1(x)}{y_2(x)}$$

$$\left. u_1'(x)\,y_1'(x) - \frac{u_1'(x)\,y_1(x)\;y_2'(x)}{y_2(x)} = f(x) \right] \cdot y_2(x)$$

$$u_1'(x)\big(y_2(x)\,y_1'(x) - y_1(x)\,y_2'(x)\big) = f(x)\,y_2(x)$$

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1(x)\,y_2'(x) - y_2(x)\,y_1'(x)$$

$$u_1'(x) = -\frac{f(x)}{W(x)}\,y_2(x)$$

$$\boxed{u_1(x) = -\int \frac{f(x)\,y_2(x)}{W(x)}\,dx\,;\quad u_2(x) = \int \frac{f(x)\,y_1(x)\,dx}{W(x)}}$$

Ex $y''(x) - 4y'(x) + 3y(x) = -e^{-2x};$

Homogeneous: $y_0'' - 4y_0' + 3y_0 = \emptyset;$

$r^2 - 4r + 3 = \emptyset$

$r_{12} = \dfrac{4 \pm \sqrt{16-12}}{2} = \dfrac{4 \pm 2}{2}$

$= 2 \pm 1 = 1, 3;$

$y_0(x) = C_1 e^x + C_2 e^{3x};$

$W(x) = \mathrm{Det}\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$

$= e^x \cdot 3e^{3x} - e^{3x} e^x = 2e^{4x}$

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x);$$

$$y_1(x) \longrightarrow C_1\, y_1(x);$$

$$y_2(x) \longrightarrow C_2\, y_2(x);$$

$$W = \mathrm{Det}\begin{bmatrix} C_1 y_1 & C_2 y_2 \\ C_1 y_1' & C_2 y_2' \end{bmatrix} = C_1 C_2 (y_1 y_2' - y_2 y_1')$$

$$u_1(x) = -\int \frac{f(x)\, C_2\, y_2(x)}{C_1 C_2 (y_1 y_2' - y_2 y_1')} \Rightarrow \frac{1}{C_1}$$

$$y_p(x) = u_1(x)\, C_1\, y_1(x) + u_2(x)\, C_2\, y_2(x)$$

$y_1(x) = e^{x}; \quad y_2 = e^{3x}$

$W(x) = 2e^{4x}$

$$u_1(x) = -\int \frac{(-)e^{-2x}\, e^{3x}}{2e^{4x}}\, dx = +\frac{1}{2}\int e^{-3x}\, dx =$$

$$= -\frac{1}{6}e^{-3x} = -\frac{e^{-3x}}{6}$$

$$u_2(x) = \int \frac{-e^{-2x}\, e^{x}}{2e^{4x}}\, dx = -\frac{1}{2}\int e^{-5x}\, dx =$$

$$= \frac{1}{10}e^{-5x}$$

$$y_r(x) = -\frac{1}{6}e^{-3x} \cdot e^{x} + \frac{1}{10}e^{-5x}\, e^{3x} = -\frac{1}{15}e^{-2x}$$

$$\frac{1}{10}^{\backslash 3} - \frac{1}{6}^{\backslash 5} = \frac{-2}{30} = -\frac{1}{15}$$

$$y'' - 4y' + 3y = -e^{-2x}$$

$$y_r = Ae^{-2x}$$

$$4Ae^{-2x} - A4 \cdot (-2)e^{-2x} + 3Ae^{-2x}$$

$$= -e^{-2x}$$

$$15Ae^{-2x} = -e^{-2x}$$

$$A = -\frac{1}{15}$$

$$y(x) = C_1e^{x} + C_2e^{3x} - \frac{1}{15}e^{-2x}$$

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$$y''(x) + y(x) = \operatorname{Tan}(x);$$

$$y_1(x) = \cos x; \quad y_2(x) = \sin(x);$$

$$W = y_1 y_2' - y_1' y_2 = \cos x \cos x + \sin x \cdot \sin x = 1$$

$$u_1(x) = -\int \frac{f(x)\, y_2(x)\, dx}{W(x)} =$$

$$= -\int \frac{\operatorname{Tan}(x) \cdot \sin x}{1}\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\, dx = -\int \frac{dx}{\cos x} + \int \cos x\, dx$$

$$\int \frac{dx}{\cos x} = +\operatorname{Log}\left[\operatorname{Tan} x + \frac{1}{\cos x}\right] \cancel{+ \sin x};$$

+here, no this term

$$u_1(x) = -\mathrm{Log}\left(\tan x + \frac{1}{\cos x}\right) + \sin x + \cancel{\sin x}$$

$$= -\mathrm{Log}\left(\tan x + \frac{1}{\cos x}\right) + \cancel{2}\sin x$$

$$u_2(x) = \int \frac{f(x)\, y_1(x)}{W(x)}\, dx = \int \frac{\tan(x)\cos x}{1}\, dx$$

$$= \int \sin x\, dx = -\cos x$$

$$y(x) = C_1 \cos x + C_2 \sin x +$$

$$- \left(\mathrm{Log}\left(\tan x + \frac{1}{\cos x}\right)\right) \cdot \cos x + \cancel{2}\sin x \cos x$$

$$- \cos(x)\sin(x) =$$

$$= C_1 \cos x + C_2 \sin x - \mathrm{Log}\left(\tan x + \frac{1}{\cos x}\right)\cos x + \cancel{\sin x \cos x}$$

$$y''(x) + y(x) = \tan x;$$

$$y''_0(x) + y_0(x) = \emptyset;$$

$$y_0(x) = e^{rx}; \quad r^2 + 1 = \emptyset; \quad r = \pm i$$

$$y(x) = C_1 e^{ix} + C_2 e^{-ix}$$

$$= C_1 \cos x + i C_1 \sin x + C_2 \cos x - i C_2 \sin x$$

$$= (\cos(x))(C_1 + C_2) + i(C_1 - C_2)\sin x$$

$$= D_1 \cos x + D_2 \sin x$$