$(\mathrm{Im}\, r)x + \varphi$ $-2-$

- $y(x) = e^{(\mathrm{Re}\, r)x} \cdot R \cos((\mathrm{Im}\, r)x - \varphi)$

$= R e^{x} \cos(5x - \varphi)$

(R, φ)

$y(\emptyset) = 1$
$y'(\emptyset) = -4$

$y'(x) = R e^{x} \cos(5x - \varphi) - R \cdot 5 \cdot e^{x} \sin(5x - \varphi)$

$y(x = \emptyset) = R \cos(\varphi) = 1$

$y'(x = \emptyset) = R \cos(\varphi) + 5 R \sin(\varphi) = -4 \Big\} R, \varphi$

$\cancel{1 + 5R =}$

$1 + 5R \sin\varphi = -4$

$\left.\begin{aligned} R \sin\varphi &= -1 \\ R \cos\varphi &= 1 \end{aligned}\right\} \Rightarrow \tan\varphi = -1$

$\varphi = -\pi/4$

$R \cos \pi/4 = 1 \Rightarrow R = 1/\sqrt{2}$

- $y(x) = e^{Re(r)x} \cdot R \cdot \sin(Im(r)x + \varphi)$

Example $y''(x) + \alpha y(x) = \emptyset$

$\alpha \gtrless \emptyset, \; \alpha = \emptyset.$

- $\alpha = \emptyset$:

  $y''(x) = \emptyset;$

- $\int\int \Rightarrow \quad y(x) = Ax + B$ $\quad A, B$ are arbitrary

- $y(x) = e^{rx}; \; r^2 = \emptyset, \; r = \emptyset;$ Double root

  $y(x) = C_1 \cdot x \cdot e^{rx} + C_2 e^{rx} =$

  $= C_1 x + C_2$

$\alpha > 0;\quad \alpha = \omega^2;$ ω is Real, positive

$y''(x) + \omega^2 y(x) = 0;$

$y(x) = e^{rx};\quad y'(x) = r e^{rx};\quad y''(x) = r^2 e^{rx};$

$(r^2 + \omega^2)\, e^{rx} = 0;$

$e^{rx} \neq 0:\quad r^2 + \omega^2 = 0;\quad r = \pm i\omega$

$y(x) = C_1 e^{i\omega x} + C_2 e^{-i\omega x}$

$y(x) = D_1 \cos \omega x + D_2 \sin \omega x;$

$y(x) = R \cos(\omega x - \varphi)$

$y(x) = R \sin(\omega x + \psi)$

$\alpha < \emptyset$

$\alpha = -\gamma^2$, γ is real, $\gamma > \emptyset$

$y''(x) - \gamma^2 y(x) = \emptyset$;

$y(x) = e^{rx}$; $r^2 - \gamma^2 = \emptyset$,

$r = \pm\gamma$;

$y(x) = C_1 e^{-x\cdot r} + C_2 e^{x\cdot r} =$

$= D_1 \sinh(x\cdot r) + D_2 \cosh(x\cdot r)$

$\sinh x = \dfrac{e^x + e^{-x}}{2}$

$\cosh x$ ———

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$y''''(x) - y(x) = 0;$$

$$y(x) = e^{rx};$$

$$r^4 - 1 = 0;$$

$$r = 1, -1, -i, i$$

$$y(x) = C_1 \cos x + C_2 \sin x$$
$$+ C_3 e^{x} + C_4 e^{-x};$$

Solution of linear nonhomogeneous equation of second order.

$$y''(x) + p(x)y'(x) + g(x)y(x) = f(x) \qquad (*)$$

general, linear, nonhomogeneous, 2nd order ode

Let $y_1(x)$ and $y_2(x)$ be solutions of $(*)$;

$$y_1''(x) + p(x)y_1'(x) + g(x)\,y_1(x) = f(x);$$

$$-$$

$$y_2''(x) + p(x)y_2'(x) + g(x)\,y_2(x) = f(x);$$

$$0 = y_1''(x) - y_2''(x) + p(x)\big(y_1'(x) - y_2'(x)\big) + g(x)\big(y_1(x) - y_2(x)\big)$$

$$\frac{d^2}{dx^2}\underbrace{(y_1(x)-y_2(x))}_{Y(x)} + p(x)\frac{d}{dx}\underbrace{(y_1(x)-y_2(x))}_{Y(x)} + q(x)\underbrace{(y_1(x)-y_2(x))}_{Y(x)} = $$

$$Y''(x) + p(x)Y'(x) + q(x)Y(x) = \cancel{0} \quad (**)$$

Linear homogeneous equation of second order

Solution to $(**)$ is a linear combination of TWO solutions to $(**)$:

$$Y(x) = C_1 Y_1(x) + C_2 Y_2(x) = y_1(x) - y_2(x)$$

where $Y_1(x), Y_2(x)$ are solutions to $(**)$

$$\underbrace{y_1(x)}_{\substack{\text{general solution} \\ \text{to } (*)}} = y_2(x) + C_1 Y_1(x) + C_2 Y_2(x)$$

Y_1, Y_2 are solutions to $(**)$

$$Y'' + p(x)Y'(x) + q(x)Y(x) = 0$$

y_1, y_2 are solutions to $(*)$

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \quad (*)$$

The general solution to
linear second order equation
nonhomogeneous

is a sum of

- a general solution to the corresponding homogeneous equation

AND

a particular solution to the nonhomogeneous equation

Method of undetermined coefficients.

Example

$$y''(x) - 2y'(x) + 26y(x) = \emptyset$$

$y(\emptyset) = 1 \qquad y'(\emptyset) = -4$

$r_1 = 1+5i; \quad r_2 = 1-5i;$

- $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

  $= C_1 e^{(1+5i)x} + C_2 e^{(1-5i)x}$ $\quad C_1, C_2$

- $y(x) = D_1 e^{(\mathrm{Re}\, r)x} \cos((\mathrm{Im}\, r)x) + D_2 e^{(\mathrm{Re}\, r)x} \sin((\mathrm{Im}\, r)x)$ $\quad D_1, D_2$

- $\cancel{y(x) = e^{\mathrm{Re}\, x}}$

  $= D_1 e^{x} \cos(5x) + D_2 e^{x} \sin(5x)$

Example

$$y''(x) + y(x) = 2e^{x}$$

Homogeneous: $y_0''(x) + y_0(x) = 0;$

$$y_0(x) = C_1 \cos x + C_2 \sin x;$$

~~Inhom~~

Particular solution

$$y_p(x) = \underset{\substack{\uparrow \\ \text{coefficient}}}{A}\, e^{x} = y'(x) = y''(x)$$

$$Ae^{x} + Ae^{x} = 2e^{x}$$

$$2A = 2, \quad A = 1; \quad \boxed{y(x) = e^{x} + C_1 \cos x + C_2 \sin x}$$

Particular Right hand sides

- exponents : $e^{\alpha x}$
- polynomial $\times$ exponent $e^{\alpha x}(a + bx + cx^2 + dx^3)$
- $\sin x$, $\cos(x)$
- polynomials $\times$ sines and cosines
- Polynomial $\times$ sines and cosines $\times$ exponents

$$f(x) = \sum_{k=1}^{n} a_k x^k \cdot e^{\alpha x} \cdot (\cos \beta x)$$

or

$$f(x) = \sum_{k=1}^{n} b_k x^k e^{\alpha x} \sin \beta x$$

$$y_p(x) = \sum_{k=1}^{m} \left(A_k x^k \sin \beta x + B_k x^k \cos \beta x\right) e^{\alpha x}$$

$m = n$

$m = n+1$

$m = n+2$

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$$y''(x) - 3y'(x) - 4y(x) = 3e^{2x} ;$$

$$y_0'' - 3y_0' - 4y_0 = \emptyset$$

$$r^2 - 3r - 4 = \emptyset$$

$$r_{12} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2}$$

$$= -1, 4$$

$$y_0(x) = C_1 e^{-x} + C_2 e^{4x}$$