$$a y''(x) + b y'(x) + c y(x) = \emptyset,$$

$$y(x) = e^{rx};$$

$$r^2 + br + c = \emptyset,$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

3 scenarios

- $b^2 > 4ac$, $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$;
  2 real roots

- $b^2 = 4ac$; $y(x) = C_1 e^{rx} + C_2 e^{rx} \cdot x$
  $r_1 = r_2 = r$

$-b^2 < 4ac,$

$$r_{1,2} = -\frac{b}{2a} \pm i\,\frac{1}{2a}\sqrt{4ac - b^2}$$

$r_1 = r_2^*$

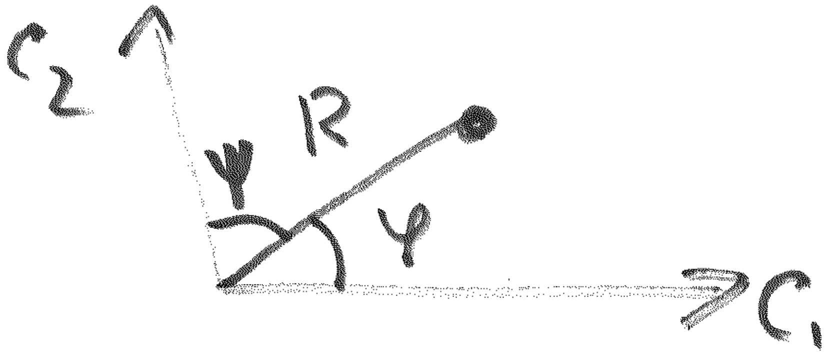
$r_1 = \bar{r}_2$

$$\boxed{\begin{aligned} \mathrm{Re}\, r &= \gamma \\ \mathrm{Im}\, r &= \omega \end{aligned}}$$

$\rightarrow$ $\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x};}$ I

$$\rightarrow y(x) = c_1 e^{\mathrm{Re}(r)x} \cdot \cos(\mathrm{Im}(r)\,x) + c_2 e^{\mathrm{Re}(r)x} \cdot \sin(\mathrm{Im}(r)\,x);$$

3



Should be ωx not (ωt)

$C_1 = R\cos\varphi$

$C_2 = R\sin\varphi$

$\cos(x-y) = \cos x\cos y + \sin x \sin y$

$$\boxed{y(x) = C_1 e^{\gamma x}\cos\omega x + C_2 e^{\gamma x}\sin\omega x} \quad \text{II}$$

$$= R\, e^{\gamma x}\cos(\omega x)\cos(\varphi) + e^{\gamma x}\sin(\omega x)\sin(\varphi)\, R$$

$\boxed{t \equiv x}$

$$= Re^{\gamma x}(\cos\omega\, \cos\varphi + \sin\omega t \sin\varphi)$$

$$\boxed{y(x) = Re^{\gamma x}\cos(\omega x - \varphi)} \quad \text{III}$$

$C_1 = R \sin \Psi$

$C_2 = R \cos \Psi$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$y(x) = C_1 e^{\gamma x} \cos \omega x + C_2 e^{\gamma x} \sin \omega x$$

$$= R e^{\gamma x} (\sin \Psi \cos \omega + \cos \Psi \sin \omega x)$$

$$= \boxed{R e^{\gamma x} \sin(\omega x + \Psi) = y(x)}$$

$t \equiv x$

Should be ωx everywhere, not ωt

Example.

$$y''(x) - 2y'(x) + 2y(x) = 0;$$

$$y(x) = e^{rx}; \quad y'(x) = re^{rx}; \quad y''(x) = r^2 e^{rx}$$

$$e^{rx} r^2 - 2e^{rx} r + 2e^{rx} = 0;$$

$$e^{rx} \neq 0, \quad x \neq \pm\infty$$

$$r^2 - 2r + 2 = 0; \quad \text{characteristic equation}$$

$$r_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$r_1 = 1 + i; \quad r_2 = 1 - i;$$

$r_1 = 1+i \quad r_2 = 1-i$

I $\quad y(x) = C_1 e^{(1+i)x} + C_2 e^{(1-i)x} \qquad (C_1, C_2)$

II $\quad y(x) = \tilde{C}_1 e^{x} \cos x + \tilde{C}_2 e^{x} \sin x \qquad \boxed{\tilde{C}_1, \tilde{C}_2}$

III $\quad y(x) = R e^{x} \cos(x - \varphi) \qquad R, \varphi$

IV $\quad y(x) = R e^{x} \sin(x + \psi) \qquad R, \psi$

Example

$$y''(x) + y'(x) + y(x) = \emptyset,$$

Find general solution

Solution

$$y(x) = e^{rx}; \quad r^2 + r + 1 = \emptyset;$$

$$r_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$$

I $\quad y(x) = C_1 e^{\left(-\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)x} + C_2 e^{\left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)x}$

II $\quad y(x) = D_1 e^{-x/2} \cos \frac{\sqrt{3}}{2} x + D_2 e^{-x/2} \sin \frac{\sqrt{3}}{2} x$

III $\quad y(x) = R e^{-x/2} \cos\left(\frac{\sqrt{3}}{2} x - \varphi\right)$

IV $\quad y(x) = R e^{-x/2} \sin\left(\frac{\sqrt{3}}{2} x + \psi\right)$

Example

$$y''(x) - 2y'(x) + 26\,y(x) = 0$$

$$y(0) = 1$$

$$y'(0) = -4$$

Solution

$$y(x) = e^{rx}; \quad r^2 - 2r + 26 = 0$$

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot 26}}{2}$$

$$= 1 \pm 5i$$

(I) $y(x) = c_1 e^{x+5ix} + c_2 e^{x-5ix}$ — 9

$y(\varnothing) = 1$

$y'(\varnothing) = -4$

$$y'(x) = c_1(1+5i)e^{(1+5i)x} + c_2(1-5i)e^{(1-5i)x}$$

$$y(\varnothing) = c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1;$$

$$y'(\varnothing) = c_1(1+5i) + c_2(1-5i) = -4$$

$$c_1(1+5i) + (1-c_1)(1-5i) + 4 = \varnothing$$

$$\underline{c_1} + 5ic_1 + 1 - 5i - \underline{c_1} + 5ic_1 + 4 = \varnothing$$

$$10ic_1 = 5i - 5 = 5(i-1)$$

$$2ic_1 = i - 1;\quad c_1 = \frac{i-1}{2i} = \frac{1}{2} + \frac{i}{2}$$

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

$$\left(f(x)g(x)\right)' = \lim_{\Delta \to 0} \frac{f(x+\Delta)g(x+\Delta) - f(x)g(x)}{\Delta}$$

$$= \frac{\left(f(x) + f'(x)\Delta + f''(x)\Delta^2/2\right)\left(g(x) + g'(x)\Delta + g''(x)\frac{\Delta^2}{2}\right)}{\Delta}$$

$$- \frac{f(x)g(x)}{\Delta}$$

$$= \frac{f(x)}{\Delta} g(x) + \Delta\left(f(x)g'(x) + f'(x)g(x)\right) + \Delta^2\left(f'g' + f''g/2 + \dots\right.$$

$$- f(x)g(x)/\Delta = f(x)g'(x) + f'(x)g(x)$$

$$c_2 = 1 - c_1 = \frac{1}{2} - \frac{i}{2} \qquad y(\emptyset) = 1$$
$$y'(\emptyset) = -4$$

$$y(x) = \left(\frac{1}{2} + \frac{i}{2}\right) e^{(1+5i)x} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(1-5i)x}$$

(11) $r_1 = 1 + 5i$; $r_2 = 1 - 5i$;

$$y(x) = D_1 e^x \cos 5x + D_2 e^x \sin 5x;$$

$$y'(x) = -5D_1 e^x \sin 5x + D_1 e^x \cos 5x + 5D_2 e^x \cos 5x + D_2 e^x \sin 5x$$

$D_1 + D_2 = 1$; $D_1 + 5D_2 = -4$; $D_1 + 5(1 - D_1) = -4$

$D_2 = 1 - D_1$ $\qquad -4D_1 = -9$

$D_1 = 9/4; \quad D_2 = 1 - D_1 = 1 - \frac{9}{4} = \frac{4}{4} - \frac{9}{4} = -\frac{5}{4}$

$y(x) = \frac{9}{4} e^x \cos 5x - \frac{5}{4} e^x \sin 5x$

III $\quad y(x) = Re^x \cos(5x - \varphi)$

IV $\quad y(x) = Re^x \sin(5x + \psi)$

$\cancel{y(0) = R = 1 \Rightarrow R = 1}$

$\cancel{y'(0)}$

$\cancel{y'(x) = e^x \sin(5x}$

$y(x) = Re^x \cos(5x - \varphi)$ 13

$y(0) = \cancel{R \cdot 1}$

$y'(x) = Re^x \cos(5x-\varphi) - 5Re^x \sin(5x-\varphi)$

$y'(0) = \cos(\varphi) + \sin(\varphi) \cdot 5 = -4$

?! $y(x) = e^x$ $\cos\varphi + \sqrt{1-\cos^2\varphi} \cdot 5 = -4$

$$y(x) = R e^{x} \cos(5x - \varphi)$$

$$y'(x) = R e^{x}\big(\cos(5x-\varphi) - 5\sin(5x-\varphi)\big)$$

$$y(0) = R\cos\varphi = 1; \quad \cos\varphi = 1/R; \quad \sin\varphi = \sqrt{1 - 1/R^2} = \frac{\sqrt{R^2-1}}{R}$$

$$y'(0) = R(\cos\varphi + 5\sin\varphi)$$

$$= R\cos\varphi - 5\sqrt{R^2-1} = 1 - 5\sqrt{R^2-1} = -4$$

$$\sqrt{R^2-1} = 1; \quad R^2 = 2, \quad R = \sqrt{2}$$

$$\cos\varphi = 1/\sqrt{2}; \quad \varphi = \pi/4$$

$$y(x) = \sqrt{2}\cos(5x - \pi/4)$$

$$y''(x) + p\,y(x) = 0$$

p is a const!

$p = 0 \quad p < 0 \quad p > 0$

$\overline{p = 0} \quad y''(x) = 0 \quad y(x) = c_1 x + c_2$

$r^2 = 0 \quad r_1 = r_2 = 0$

$$y(x) = c_1 e^{0x} + c_2 x e^{0x}$$

$$= c_1 + c_2 x$$